

STABLY CAYLEY GROUPS OVER FIELDS OF CHARACTERISTIC 0

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ABSTRACT. A linear algebraic group G is called a Cayley group if it admits a Cayley map, i.e., a G -equivariant birational isomorphism between the group variety G and the Lie algebra $\mathrm{Lie}(G)$. A Cayley map can be thought of as a partial algebraic analogue of the exponential map. A prototypical example is the classical Cayley map for the special orthogonal group \mathbf{SO}_n defined by A. Cayley in 1846. A k -group G is called *stably Cayley* if $G \times_k \mathbb{G}_m^r$ is Cayley for some $r \geq 0$. Over an algebraically closed field of characteristic 0, Cayley and stably Cayley simple groups were classified by N. Lemire, V. L. Popov and Z. Reichstein in 2006.

In this paper we study reductive Cayley groups in the case where k is an arbitrary field of characteristic zero. The condition of being Cayley is considerably more delicate in this setting. For example, an algebraic k -torus is Cayley if and only if it is k -rational. Our main results are a criterion for a reductive k -group G to be stably Cayley, formulated in terms of its character lattice, and the classification of stably Cayley simple (but not necessarily absolutely simple) groups.

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1. INTRODUCTION

Let k be a field of characteristic 0 and G be a connected linear algebraic k -group. We say that a birational isomorphism $\phi: G \dashrightarrow \mathrm{Lie}(G)$ is a *Cayley map* if it is equivariant with respect to the conjugation action of G on itself and the adjoint action of G on its Lie algebra $\mathrm{Lie}(G)$, respectively.

A Cayley map can be thought of as a (partial) algebraic analogue of the exponential map. A prototypical example is the classical Cayley map for the special orthogonal group \mathbf{SO}_n defined by A. Cayley in 1846 [Ca]; cf. [LPR, Introduction]. We say that G is a *Cayley group* if it admits a Cayley map. We say that G is *stably Cayley* if $G \times_k \mathbb{G}_m^r$ is Cayley for some $r \geq 0$, where \mathbb{G}_m denotes the multiplicative group. In the case where k is algebraically closed, Cayley and stably Cayley groups were studied in [LPR].

Example 1.1. If k is algebraically closed and G be a reductive k -group then by [LPR, Theorem 1.27] G is stably Cayley if and only if its character lattice $X(G)$ is quasi-permutation; see Definition 2.1. Here $X(G)$ is the character lattice of a maximal torus T of G , with the natural action of the Weyl group $W = W(G, T)$ of G .

In this paper we will focus on the case where k is an arbitrary field of characteristic 0. The following “toy” example illustrates how much more intricate the notions of Cayley and stably Cayley group become in this situation.

Example 1.2. Let T be a k -torus of dimension d . By definition, T is Cayley (respectively, stably Cayley) over k if and only if T is k -rational (respectively, stably k -rational). If k is algebraically closed, then $T \simeq \mathbb{G}_m^d$, hence T is always rational, and thus always Cayley.

There is a well-known criterion for stable rationality of T in terms of its character lattice $X(T)$ [V2, Theorem 4.7.2]: T is stably rational if and only if the character lattice $X(T)$ is quasi-permutation (see Definition 2.1). Note that the term “character lattice” has a different meaning here, compared to Example 1.1: it is the lattice of characters of T with the natural action of the absolute Galois group $\text{Gal}(k)$.

It has been conjectured that every stably rational torus is rational. To the best of our knowledge, this conjecture is still open, and there is no simple lattice-theoretic criterion for the rationality of T .

Our first main result, Theorem 1.3 below, is a common generalization of Examples 1.1 and 1.2. We begin by defining the *character lattice* of a reductive k -group G in a way that bridges the special cases considered in these two examples. Let $T \subset G$ be a maximal torus of G and \bar{k} be a fixed algebraic closure of k . We set $\bar{G} = G \times_k \bar{k}$, $\bar{T} = T \times_k \bar{k}$. Consider the character group $X(\bar{T})$ of \bar{T} and the Weyl group $W = W(\bar{G}, \bar{T}) \subset \text{Aut } X(\bar{T})$. Choose a Borel subgroup $\bar{B} \subset \bar{G}$ containing \bar{T} . This choice of \bar{B} defines a basis Δ of the root system $R = R(\bar{G}, \bar{T})$. We obtain a homomorphism (*-action)

$$\text{Gal}(\bar{k}/k) \rightarrow \text{Aut}(X(\bar{T}), R, \Delta) \subset \text{Aut } X(\bar{T}),$$

see [T, §2.3], or [Sp1, §3.2], or §3.4 below for details. Let A_Ψ denote the image of this homomorphism in $\text{Aut } X(\bar{T})$, it is nontrivial if and only if G is an outer form (of a split group). The finite group A_Ψ normalizes $W \subset \text{Aut } X(\bar{T})$, and we can define a group $\Upsilon := W \rtimes A_\Psi \subset \text{Aut } X(\bar{T})$. The pair $(\Upsilon, X(\bar{T}))$ does not depend on the choice of T and B (up to an isomorphism), see §3.6 below. We say that $(\Upsilon, X(\bar{T}))$ is the *character lattice* $X(G)$ of G .

Equivalently, $X(G)$ is the character lattice of the generic torus T_{gen} of G ; this torus is defined over a certain transcendental field extension K_{gen} of k , see [V2, §4.2] for details.

Informally speaking, we think of the Weyl group W as “the geometric part” of Υ , and of the image A_Ψ of $\text{Gal}(\bar{k}/k)$ under the *-action as “the algebraic part”. In this sense Examples 1.1 and 1.2 represent two opposite extremes, where the group Υ is “purely geometric” and “purely algebraic”, respectively. As we pass from a reductive group G to its generic torus T_{gen} , the geometric part migrates to the algebraic part, while the overall group Υ remains the same (but now entirely algebraic).

With these preliminaries out of the way, we are ready to state our first main theorem.

Theorem 1.3. *Let G be a reductive k -group. The following are equivalent:*

- (a) G is stably Cayley;
- (b) for every field extension K/k , every maximal K -torus $T \subset G_K$ is stably rational over K ;
- (c) the generic K_{gen} -torus T_{gen} of G is stably rational;

(d) *the character lattice $X(G) = (\Upsilon, X(\overline{T}))$ of G is quasi-permutation.*

Next we turn our attention to classifying stably Cayley simple groups. For an algebraically closed base field k this was done in [LPR, Theorem 1.28]. In this paper we will use Theorem 1.3 to prove the following classification of stably Cayley simple groups over an *arbitrary* field of characteristic 0.

Theorem 1.4. *Let k be a field of characteristic 0 and G be an absolutely simple k -group. Then the following conditions are equivalent:*

- (a) *G is stably Cayley over k ;*
- (b) *G is any k -form of one of the following groups:*

SL_3 , PGL_n ($n = 2$ or $n \geq 3$ odd), SO_n ($n \geq 5$), Sp_{2n} ($n \geq 1$), \mathbf{G}_2 ,
or an inner k -form of PGL_n ($n \geq 4$ even).

Theorem 1.5. *Let G be a simple (but not necessarily absolutely simple) k -group over a field k of characteristic 0. Then the following conditions are equivalent:*

- (a) *G is stably Cayley over k ;*
- (b) *G is isomorphic to $R_{l/k}(G_0)$, where l/k is a finite field extension and G_0 is either a stably Cayley absolutely simple group over l (i.e., one of the groups listed in Theorem 1.4(b)) or an outer l -form of SO_4 .*

Here $R_{l/k}$ denotes the Weil functor of restriction of scalars.

A key consequence of Theorem 1.3 is that, for a reductive k -group G , being *stably Cayley* is a property of its character lattice. If k is algebraically closed, then the property of being Cayley (and not just stably Cayley) also depends only on the character lattice of G ; see [LPR, §3.1]. However, if k is not assumed to be algebraically closed, the following example shows that this is no longer the case.

Example 1.6. Let T be a 2-dimensional k -torus such that the image of the natural map $\mathrm{Gal}(k) \rightarrow \mathrm{Aut}(X(\overline{T})) = \mathrm{GL}_2(\mathbb{Z})$ is isomorphic to $W(\mathbf{G}_2) = \mathrm{S}_3 \times \mathrm{S}_2$. Note that such tori exist over some fields k of characteristic 0; see [V2, Example 4.9.7]. Let G be a split group of type \mathbf{G}_2 over k .

Then T and G have the isomorphic character lattices; both T and G are stably Cayley. Since every 2-dimensional k -torus is k -rational (see [V2, Example 4.9.7]), T is Cayley over k (cf. Example 1.2). On the other hand, by a theorem of V.A. Iskovskikh, \mathbf{G}_2 is not Cayley over k (not even over \bar{k}); see [LPR, Proposition 9.10]. \square

Informally speaking, the character lattice of T is “purely arithmetic” and the character lattice of \mathbf{G}_2 is “purely geometric”. Theorem 1.3 tells us that this distinction does not matter if we only want to know whether or not our groups are stably Cayley. Example 1.6 tells us that it does matter when “stably Cayley” is replaced by “Cayley”.

We do not know if there are similar examples where both groups are semisimple. Nevertheless, the problem of classifying Cayley simple groups,

in a manner analogous to Theorems 1.4 and 1.5, appears to be out of reach at the moment. In particular, we do not know which (if any) outer forms of \mathbf{PGL}_n are Cayley for any odd integer $n \geq 3$.

The rest of this paper is structured as follows. §2, §3 and §4 are devoted to preliminary material on quasi-permutation lattices, algebraic groups over non-algebraically closed fields, and (G, S) -fibrations, respectively. These results are then used in the proof of Theorem 1.3 in §5. Theorem 1.4 is an easy consequence of Theorem 1.3 and previously known results on character lattices of absolutely simple groups from [CK] and [LPR]; the details of this (short) argument are presented in §6.

The proof of Theorem 1.5 relies on new results of character lattices and thus requires considerably more work. After passing to the algebraic closure \bar{k} , we are faced with the problem of classifying semisimple stably Cayley groups of the form $G = H^m/C$, where H is a simply connected simple group over \bar{k} and $C \subset H^m$ is a central subgroup. Our classification theorem for such groups is stated in §7; see Theorem 7.1. The proof of Theorem 7.1, based on case-by-case analysis, occupies §§8–15. In §16 we deduce Theorem 1.5 from Theorem 7.1 by passing back from \bar{k} to k .

2. PRELIMINARIES ON QUASI-PERMUTATION LATTICES

Let Γ be a finite group. By a Γ -lattice we mean a finitely generated free abelian group M viewed together with an integral representation $\Gamma \rightarrow \text{Aut}(M)$. We also think of M as of a $\mathbb{Z}[\Gamma]$ -module; by a morphism (or exact sequence) of lattices we mean a morphism (or exact sequence) of $\mathbb{Z}[\Gamma]$ -modules. When we write just “lattice”, rather than “ Γ -lattice”, we mean a Γ -lattice for some finite group Γ .

Now let k be a field, $T_{\text{spl}} = \mathbb{G}_m^d$ be the split d -dimensional k -torus, Γ be a finite group. By a multiplicative action of a finite group Γ on T_{spl} we mean an action by automorphisms of T_{spl} as an algebraic group over k . Recall that the following objects are in a natural bijective correspondence:

- (i) Γ -lattices of rank d (up to isomorphism);
- (ii) integral representations $\phi: \Gamma \rightarrow \text{GL}_d(\mathbb{Z})$ (up to conjugacy in $\text{GL}_d(\mathbb{Z})$);
- (iii) multiplicative actions $\Gamma \rightarrow \text{Aut}_{\text{grp}}(T_{\text{spl}})$ (up to an automorphism of T_{spl} as an algebraic group).

A Γ -lattice L is called *permutation* if it has a \mathbb{Z} -basis permuted by Γ . Two Γ -lattices L and L' are called *equivalent*, written $L \sim L'$, if there exist short exact sequences

$$0 \rightarrow L \rightarrow E \rightarrow P \rightarrow 0 \quad \text{and} \quad 0 \rightarrow L' \rightarrow E \rightarrow P' \rightarrow 0$$

with the same Γ -lattice E , where P and P' are permutation Γ -lattices. For a proof that this is indeed an equivalence relation, see [CS1, Lemma 8]. Note that if there exists a short exact sequence

$$0 \rightarrow L \rightarrow L' \rightarrow Q \rightarrow 0,$$

where Q is a permutation Γ -lattice, then the trivial short exact sequence

$$0 \rightarrow L' \rightarrow L' \rightarrow 0 \rightarrow 0$$

shows that $L \sim L'$. In particular, if P is a permutation Γ -lattice, then the short exact sequence

$$0 \rightarrow 0 \rightarrow P \rightarrow P \rightarrow 0$$

shows that $P \sim 0$. Note that if Γ -lattices L, L', M, M' satisfy $L \sim L'$ and $M \sim M'$ then $L \oplus M \sim L' \oplus M'$.

Definition 2.1. A Γ -lattice L is called *quasi-permutation* if it is equivalent to a permutation lattice, i.e., if there exists a short exact sequence

$$0 \rightarrow L \rightarrow P \rightarrow P' \rightarrow 0,$$

where both P and P' are permutation Γ -lattices.

We say that a faithful Γ -action on an algebraic variety X , defined over k , is *linearizable* (respectively, *stably linearizable*) if X is Γ -equivariantly birationally isomorphic (respectively, Γ -equivariantly stably birationally isomorphic) to a finite-dimensional k -vector space V with a linear Γ -action.

Remark 2.2. By the no-name lemma any two faithful linear actions of a finite group Γ on k -vector spaces V_1 and V_2 are stably Γ -equivariantly birationally equivalent; see, e.g., [LPR, Lemma 2.12(c)]. This makes stable linearizability a particularly natural notion.

Lemma 2.3. *Let L be a Γ -lattice, and let T_L be the associated split k -torus with multiplicative Γ -action (i.e. $X(T_L) = L$).*

- (a) *If L is a permutation lattice then the Γ -action on T_L is linearizable.*
- (b) *L is quasi-permutation if and only if the Γ -action on T_L is stably linearizable.*

Proof. (a) Suppose $L \simeq \mathbb{Z}[S]$ for some finite Γ -set S . Let V be the k -vector space with basis $(e_s)_{s \in S}$. Then V carries a natural (permutation) Γ -action. The morphism $T \rightarrow V$ given by

$$t \rightarrow \sum_{s \in S} s(t)e_s$$

is easily seen to be a Γ -equivariant birational isomorphism.

(b) Let P be a faithful permutation Γ -lattice (e.g., $P = \mathbb{Z}[\Gamma]$). Let V be the linear representation of G constructed in part (a). It now suffices to show that the following conditions are equivalent:

- (i) L is quasi-permutation,
- (ii) $L \sim P$,
- (iii) T_L and T_P are Γ -equivariantly stably birationally isomorphic,
- (iv) T_L and V are Γ -equivariantly stably birationally isomorphic,
- (v) T_L is stably linearizable.

Indeed, (i) and (ii) are equivalent by Definition 2.1. (ii) and (iii) are equivalent by [LL, Proposition 1.4]; note that, in the terminology of [LL, §1.4] $k(L)$ is precisely the field of rational functions of T_L .

In the proof of part (a) we showed that T_P and V are Γ -equivariantly birationally isomorphic. Consequently, (iii) is equivalent to (iv). Finally, (iv) \implies (v) by definition, and (v) \implies (iv) by the no-name lemma; see Remark 2.2. \square

Lemma 2.4 (cf. [LPR], Proposition 4.8). *Let W_1, \dots, W_m be finite groups. For each $i = 1, \dots, m$, let V_i be a finite-dimensional \mathbb{Q} -representation of W_i . Set $V := V_1 \oplus \dots \oplus V_m$. Suppose $L \subset V$ be a free abelian subgroup, invariant under $W := W_1 \times \dots \times W_m$.*

If L is a quasi-permutation W -lattice, then $L_i := L \cap V_i$ is a quasi-permutation W_i -lattice, for each $i = 1, \dots, m$.

Proof. It suffices to prove the lemma for $i = 1$. Set $V' := V/V_1 = V_2 \oplus \dots \oplus V_m$ and $L' = L/L_1 \subset V'$. Then W_1 acts trivially on V' and on L' , in particular, L' is a permutation W_1 -lattice. It follows from the short exact sequence of W_1 -lattices

$$0 \rightarrow L_1 \rightarrow L \rightarrow L' \rightarrow 0$$

that the W_1 -lattices L_1 and L are equivalent.

Now assume that L is a quasi-permutation W -lattice. Then it is a quasi-permutation W_1 -lattice, and hence so is L_1 . \square

Lemma 2.5 (cf. [LPR], Lemma 4.7). *Let W_1, \dots, W_m be finite groups. For each $i = 1, \dots, m$, let L_i be a W_i -lattice. Set $W := W_1 \times \dots \times W_m$ and construct a W -lattice $L := L_1 \oplus \dots \oplus L_m$.*

Then L is a quasi-permutation W -lattice if and only if L_i is a quasi-permutation W_i -lattice for each $i = 1, \dots, m$.

Proof. The “if” assertion is obvious from the definition. The “only if” assertion follows from Lemma 2.4. \square

Lemma 2.6. *Let Γ be a finite group and L a Γ -lattice of rank 1 or 2. Then L is quasi-permutation.*

Proof. This is easily deduced from [V2, §4.9, Examples 6, 7]. \square

3. ALGEBRAIC GROUPS, THEIR AUTOMORPHISMS AND CHARACTER LATTICES

3.1. Let G be a *split* connected reductive group over a field k . Let $T \subset G$ be a split maximal torus of G . Let $\Psi(G, T)$ be the *root datum* of (G, T) . This means that

$$\Psi(G, T) = (X, X^\vee, R, R^\vee),$$

where $X = X(T)$ is the character group of T , $X^\vee = \text{Hom}(X, \mathbb{Z})$ is the cocharacter group of T , $R = R(G, T) \subset X$ is the root system of G with

respect to T , and $R^\vee \subset X^\vee$ is the coroot system of G with respect to T , see [Sp1, §1.1] or [Sp2, §7.4] for details.

Let $B \subset G$ be a Borel subgroup containing T . Let $\Xi(G, T, B)$ be the *based root datum* of (G, T, B) . This means that

$$\Xi(G, T, B) = (X, X^\vee, R, R^\vee, \Delta, \Delta^\vee),$$

where X, X^\vee, R, R^\vee are as above, $\Delta \subset R$ is the basis of R defined by B , and $\Delta^\vee \subset R^\vee$ is the corresponding basis of R^\vee , see [Sp1, §1.9] for details.

Let $Z = Z(G)$ be the center of G . We set $G^{\text{ad}} = G/Z$, $T^{\text{ad}} = T/Z \subset G^{\text{ad}}$. We identify G^{ad} with the algebraic group of inner automorphisms $\text{Inn}(G)$ of G . If $g \in G^{\text{ad}}(k)$ (or $g \in T^{\text{ad}}(k)$), we write $\text{inn}(g)$ for the corresponding inner automorphism of G . The group $G^{\text{ad}}(k)$ acts on the set of pairs (T, B) by $(g, (T, B)) \mapsto (\text{inn}(g)(T), \text{inn}(g)(B))$. It is well known that this action is transitive and the stabilizer in $G^{\text{ad}}(k)$ of a pair (T, B) is $T^{\text{ad}}(k)$.

We define the *canonical based root datum* $\Xi(G)$ as follows (see also [Ko, 1.1, 1.2]). For any pair (T, B) as above we consider $\Xi(G, T, B)$. If (T', B') is another such pair, then there exists an element $g \in G^{\text{ad}}(k)$ such that $(T', B') = \text{inn}(g)(T, B)$, and so we obtain an induced isomorphism

$$\text{inn}(g)^*: \Xi(G, T', B') \xrightarrow{\sim} \Xi(G, T, B).$$

This isomorphism does not depend on the choice of g (because the coset Tg is uniquely determined), and therefore allows us to identify $\Xi(G, T, B)$ and $\Xi(G, T', B')$. We identify the based root data $\Xi(G, T, B)$ for all such pairs (T, B) and obtain a canonical based root datum $\Xi(G)$. We write $\Xi(G) = (X, X^\vee, R, R^\vee, \Delta, \Delta^\vee)$. Then, in particular, we obtain the canonical root system $R(G)$, the canonical Weyl group $W(G) := W(R(G))$, etc.

We define a canonical homomorphism

$$\varphi_\Psi: \text{Aut } \Psi(G, T) \rightarrow \text{Aut } \Xi(G)$$

as follows.

Choose a Borel subgroup $B \subset G$ containing T , it defines a basis Δ of $R = R(G, T)$. Let $v \in \text{Aut } \Psi(G, T)$, then the set $v(\Delta)$ is a basis of R . Since the Weyl group $W = W(R)$ acts simply transitively on the set of bases of R (cf. [Bou, §VI.1.5, Thm. 2(i)]), there exists a unique $w \in W$ such that

$$(3.1) \quad w(v(\Delta)) = \Delta.$$

We obtain an automorphism

$$\varphi_\Psi(v) := w \circ v: \Xi(G, T, B) \rightarrow \Xi(G, T, B).$$

Thus we obtain a homomorphism

$$\varphi_\Psi: \text{Aut } \Psi(G, T) \rightarrow \text{Aut } \Xi(G, T, B) = \text{Aut } \Xi(G).$$

We show that this homomorphism does not depend on the choice of a Borel subgroup B containing T . Let $B' \subset G$ be another Borel subgroup containing T , it defines a basis Δ' of R and a homomorphism

$$\varphi'_\Psi: \text{Aut } \Psi(G, T) \rightarrow \text{Aut } \Xi(G, T, B').$$

Choose $n \in N_G(T)$ such that $nBn^{-1} = B'$, then $\Delta = n^*(\Delta')$. From (3.1) we obtain

$$\begin{aligned} (w \circ v \circ n^*)(\Delta') &= n^*(\Delta'), \\ (n^{*-1} \circ w \circ v \circ n^*)(\Delta') &= \Delta', \\ n^{*-1} \varphi_\Psi(v) n^* &= \varphi'_\Psi(v). \end{aligned}$$

Since we identify $\Xi(G, T, B)$ and $\Xi(G, T, B')$ via n^* , we see that φ_Ψ and φ'_Ψ give the same homomorphism $\text{Aut } \Psi(G, T) \rightarrow \text{Aut } \Xi(G)$.

We have shown that the homomorphism φ_Ψ does not depend on the choice of a Borel subgroup B containing T . If we choose such a Borel subgroup, we obtain a splitting (homomorphic section) of φ_Ψ

$$\psi_\Psi: \text{Aut } \Xi(G) \rightarrow \text{Aut } \Psi(G, T),$$

because $\text{Aut } \Xi(G, T, B) \subset \text{Aut } \Psi(G, T)$. Thus we obtain a split short exact sequence

$$1 \rightarrow W \rightarrow \text{Aut } \Psi(G, T) \rightarrow \text{Aut } \Xi(G) \rightarrow 1.$$

Let $A \subset \text{Aut } \Xi(G)$ be a finite subgroup. Set $\Upsilon = \varphi_\Psi^{-1}(A)$, then $W \subset \Upsilon \subset \text{Aut } \Psi(G, T)$. We have a split exact sequence

$$1 \rightarrow W \rightarrow \Upsilon \rightarrow A \rightarrow 1.$$

This exact sequence does not depend on the choice of a Borel subgroup $B \supset T$. After we choose such a subgroup B , we may set $A_\Psi = \psi_\Psi(A) \subset \Upsilon$. We have

$$W \cap A_\Psi = 1, \quad W \cdot A_\Psi = \Upsilon.$$

The group A_Ψ acts on W by conjugation, and we may write

$$\Upsilon = W \rtimes A_\Psi.$$

3.2. Let \overline{G} be a connected reductive group over an algebraic closure \bar{k} of k . We denote by $\text{SAut}(\overline{G})$ the group of \bar{k}/k -semi-automorphisms of \overline{G} . For a definition of semi-automorphisms see [Brv, §1.1] (where they are called semialgebraic automorphisms) or [FSS, §1.2] (where they are called semilinear automorphisms). If G is a k -form of \overline{G} , then any element $\sigma \in \text{Gal}(\bar{k}/k)$ defines a σ -semi-automorphism $\sigma_*: \overline{G} \rightarrow \overline{G}$, and any semi-automorphism of \overline{G} is of the form $a = \alpha \circ \sigma_*$ where $\sigma \in \text{Gal}(\bar{k}/k)$ and $\alpha: \overline{G} \rightarrow \overline{G}$ is a \bar{k} -automorphism of the \bar{k} -group \overline{G} .

Since the group $\text{SAut}(\overline{G})$ acts on \overline{G} , it acts on the canonical based root datum $\Xi(\overline{G})$. We will now describe this action explicitly. Fix $(\overline{T}, \overline{B})$ as above. Let $a \in \text{SAut}(\overline{G})$, then there exists $g \in \overline{G}^{\text{ad}}(\bar{k})$ such that $\text{inn}(g)^{-1}(a(\overline{T}), a(\overline{B})) = (\overline{T}, \overline{B})$. Set $a_{\overline{T}, \overline{B}} = \text{inn}(g)^{-1} \circ a$, then

$$a = \text{inn}(g) \circ a_{\overline{T}, \overline{B}},$$

where $g \in \overline{G}^{\text{ad}}(\bar{k})$ and $a_{\overline{T}, \overline{B}}$ stabilizes the pair $(\overline{T}, \overline{B})$. The semi-automorphism $a_{\overline{T}, \overline{B}}$ of \overline{G} defines a semi-automorphism of \overline{T} depending only on a (since the coset $\overline{T}^{\text{ad}} g^{-1}$ is uniquely determined). Denote by $\varphi(a)$ the inverse of the automorphism of $X = \mathbf{X}(\overline{T})$ induced by $a_{\overline{T}, \overline{B}}$, then $\varphi(a)$ takes $R = R(\overline{G}, \overline{T})$ to itself and permutes the elements of the basis Δ of R defined by \overline{B} (since $a_{\overline{T}, \overline{B}}$ takes \overline{B} to itself). We obtain an automorphism

$$\varphi(a): \Xi(\overline{G}, \overline{T}, \overline{B}) \rightarrow \Xi(\overline{G}, \overline{T}, \overline{B}),$$

depending only on a . Thus we obtain a homomorphism

$$(3.2) \quad \varphi: \text{SAut}(\overline{G}) \rightarrow \text{Aut } \Xi(\overline{G}, \overline{T}, \overline{B}) = \text{Aut } \Xi(\overline{G}).$$

Let $(\overline{T}', \overline{B}')$ be another pair as above. We may write

$$(\overline{T}', \overline{B}') = \text{inn}(u)(\overline{T}, \overline{B}),$$

where $u \in \overline{G}^{\text{ad}}(\bar{k})$. We write

$$\begin{aligned} a &= \text{inn}(g) a_{\overline{T}, \overline{B}} = \text{inn}(g) \circ (a_{\overline{T}, \overline{B}} \text{inn}(u) a_{\overline{T}, \overline{B}}^{-1}) \circ \text{inn}(u)^{-1} \circ \text{inn}(u) a_{\overline{T}, \overline{B}} \text{inn}(u)^{-1} \\ &= \text{inn}(g a_{\overline{T}, \overline{B}}(u) u^{-1}) \circ \text{inn}(u) a_{\overline{T}, \overline{B}} \text{inn}(u)^{-1}. \end{aligned}$$

Set $g' = g a_{\overline{T}, \overline{B}}(u) u^{-1}$ and

$$(3.3) \quad a_{\overline{T}', \overline{B}'} = \text{inn}(u) a_{\overline{T}, \overline{B}} \text{inn}(u)^{-1}.$$

Then $a = \text{inn}(g') \circ a_{\overline{T}', \overline{B}'}$ and $a_{\overline{T}', \overline{B}'}(\overline{T}', \overline{B}') = (\overline{T}', \overline{B}')$. We denote by $\varphi'(a)$ the inverse of the automorphism of $X' = \mathbf{X}(\overline{T}')$ induced by $a_{\overline{T}', \overline{B}'}$, then we obtain a homomorphism

$$\varphi': \text{SAut}(\overline{G}) \rightarrow \text{Aut } \Xi(\overline{G}, \overline{T}', \overline{B}') = \text{Aut } \Xi(\overline{G}).$$

It follows from (3.3) that the homomorphisms φ and φ' coincide (because we identify $\Xi(\overline{G}, \overline{T}, \overline{B})$ and $\Xi(\overline{G}, \overline{T}', \overline{B}')$ via $\text{inn}(u)$).

3.3. Let a triple (G, T, B) be as in 3.1, i.e. G is a *split* connected reductive k -group, $T \subset G$ a split k -torus, and $B \subset G$ a Borel subgroup containing T . Composing the embedding $\text{Aut}(G) \hookrightarrow \text{SAut}(\overline{G})$ with the homomorphism (3.2), we obtain a canonical homomorphism

$$\lambda: \text{Aut}(G) \rightarrow \text{Aut } \Xi(\overline{G}) = \text{Aut } \Xi(G).$$

By a *pinning* of (G, T, B) we mean a choice of a nonzero $X_\alpha \in \mathfrak{g}_\alpha$ for each $\alpha \in \Delta$, where

$$\text{Lie}(G) = \text{Lie}(T) \oplus \bigoplus_{\alpha \in R} \mathfrak{g}_\alpha$$

is the root decomposition, and Δ is the basis of $R = R(G, T)$ associated with B . By the isomorphism theorem, see [SGA3, Exposé XXIII, Thm. 4.1] or [Co, Proposition 1.5.5], there is a canonical isomorphism

$$\text{Aut } \Xi(G) \xrightarrow{\sim} \text{Aut}(G, T, B, (X_\alpha)_{\alpha \in \Delta}),$$

which gives a splitting (homomorphic section) $\psi: \text{Aut } \Xi(G) \rightarrow \text{Aut}(G)$ of the homomorphism λ . We obtain a split exact sequence

$$1 \rightarrow \text{Inn}(G) \rightarrow \text{Aut}(G) \xrightarrow{\lambda} \text{Aut } \Xi(G) \rightarrow 1.$$

Let $A \subset \text{Aut } \Xi(G)$ be a finite subgroup. Set $M = \lambda^{-1}(A) \subset \text{Aut}(G)$. We have a split exact sequence

$$1 \rightarrow \text{Inn}(G) \rightarrow M \rightarrow A \rightarrow 1.$$

Choose T, B , and a pinning of (G, T, B) , then we set $A_G = \psi(A) \subset M$. We have

$$\text{Inn}(G) \cap A_G = 1, \text{Inn}(G) \cdot A_G = M.$$

The group A_G acts on $\text{Inn}(G)$ by conjugation, and we have

$$M = \text{Inn}(G) \rtimes A_G.$$

We identify $\Xi(G, T, B)$ with $\Xi(G)$ and denote by

$$A_\Psi \subset \text{Aut } \Xi(G, T, B) \subset \text{Aut } \Psi(G, T)$$

the subgroup corresponding to $A \subset \text{Aut } \Xi(G)$. Let $L = \text{Stab}_M(T) = N^{\text{ad}} \rtimes A_G$, where $N^{\text{ad}} = \text{Stab}_{G^{\text{ad}}}(T)$. Then $L \supset T^{\text{ad}} = T/Z(G)$. The group L acts on the pair (G, T) , and L/T^{ad} acts on $\Psi(G, T)$, and these actions induce an isomorphism $A_G \xrightarrow{\sim} A_\Psi$. We have

$$L/T^{\text{ad}} = (N^{\text{ad}} \rtimes A_G)/T^{\text{ad}} = (N^{\text{ad}}/T^{\text{ad}}) \rtimes A_\Psi = W \rtimes A_\Psi.$$

3.4. Let G be a connected reductive k -group, not necessarily split. Then any $\sigma \in \text{Gal}(\bar{k}/k)$ defines a σ -semi-automorphism $\sigma_*: \bar{G} \rightarrow \bar{G}$, so we obtain a canonical homomorphism

$$\text{Gal}(\bar{k}/k) \rightarrow \text{SAut}(\bar{G}).$$

Composing this homomorphism with (3.2), we obtain a canonical homomorphism

$$\beta: \text{Gal}(\bar{k}/k) \rightarrow \text{Aut } \Xi(\bar{G}).$$

We denote by A the image of β , it is a finite subgroup of $\text{Aut } \Xi(\bar{G})$.

We write $\Xi(\bar{G}) = (X, X^\vee, R, R^\vee, \Delta, \Delta^\vee)$, then we set $\Psi(\bar{G}) = (X, X^\vee, R, R^\vee)$. We have the Weyl group $W = W(R) \subset \text{Aut } \Psi(\bar{G})$, and we have a subgroup $\text{Aut } \Xi(\bar{G}) \subset \text{Aut } \Psi(\bar{G})$. Since any element of $\text{Aut } \Xi(\bar{G})$ takes Δ to itself, and $\text{Stab}_W(\Delta) = \{e\}$, we have $W \cap \text{Aut } \Xi(\bar{G}) = \{e\}$. Since W is a normal subgroup of $\text{Aut } \Psi(\bar{G})$, the group $\text{Aut } \Xi(\bar{G})$ normalizes W and acts on it. We have

$$\text{Aut } \Psi(\bar{G}) = W \cdot \text{Aut } \Xi(\bar{G}) = W \rtimes \text{Aut } \Xi(\bar{G}).$$

It follows that $W \cap A = \{e\}$, A normalizes W and acts on it. We set

$$\Upsilon = W \cdot A = W \rtimes A,$$

then $\Upsilon \subset \text{Aut } \Psi(\bar{G}) \subset \text{Aut}(X)$.

Definition 3.5. For a connected reductive k -group G with canonical root datum $\Xi(\overline{G}) = (X, X^\vee, R, R^\vee, \Delta, \Delta^\vee)$ we call the *character lattice* of G the lattice $X(G) := X$ with the action of the finite group $\Upsilon = W \rtimes A$ as above.

3.6. We write the definition of the character lattice of a connected reductive k -group G in a simple language.

We start from a maximal torus $T \subset G$, defined over k . We have the usual action of the Galois group

$$\rho_T: \text{Gal}(\bar{k}/k) \rightarrow \text{Aut } \Psi(\overline{G}, \overline{T}) \subset \text{Aut}(X(\overline{T})),$$

where $\Psi(\overline{G}, \overline{T}) = (X, X^\vee, R, R^\vee)$. Since $W := W(\overline{G}, \overline{T})$ is a normal subgroup of $\text{Aut } \Psi(\overline{G}, \overline{T})$, the product

$$\Upsilon_T := W \cdot \text{im } \rho_T$$

is a subgroup of $\text{Aut } \Psi(\overline{G}, \overline{T})$ and, hence, of $\text{Aut}(X(\overline{T}))$.

Now choose a Borel subgroup $\overline{B} \supset \overline{T}$ in \overline{G} , it defines a basis $\Delta \subset R(\overline{G}, \overline{T})$. For any $\sigma \in \text{Gal}(\bar{k}/k)$ there exists a unique $w \in W$ such that $w(\rho_T(\sigma)(\Delta)) = \Delta$. Set $\beta(\sigma) = w \circ \rho_T(\sigma) \in \text{Aut}(X(\overline{T}))$, then we obtain a homomorphism

$$\beta = \beta_{T, \overline{B}}: \text{Gal}(\bar{k}/k) \rightarrow \text{Aut}(X(\overline{T})).$$

Set

$$\Upsilon_{T, \overline{B}} := W \cdot \text{im } \beta_{T, \overline{B}} = W \rtimes \text{im } \beta_{T, \overline{B}} \subset \text{Aut } \Psi(\overline{G}, \overline{T}).$$

It follows from the definitions that

$$\Upsilon_{T, \overline{B}} = \Upsilon_T,$$

in particular, the group $\Upsilon_{T, \overline{B}} \subset \text{Aut } X(\overline{T})$ does not depend on the choice of a Borel subgroup $\overline{B} \supset \overline{T}$.

Now let $T' \subset G$ be another maximal torus defined over k . Choose a Borel subgroup $\overline{B}' \supset \overline{T}$. There exists an element $u \in G^{\text{ad}}(\bar{k})$ such that $\text{inn}(u)(\overline{T}, \overline{B}) = (\overline{T}', \overline{B}')$. We obtain an isomorphism

$$\text{inn}(u)_*: \text{Aut } X(\overline{T}) \rightarrow \text{Aut } X(\overline{T}'),$$

that clearly takes $W = W(\overline{G}, \overline{T})$ to $W' := W(\overline{G}, \overline{T}')$. It follows from (3.3) that $\text{inn}(u)_*$ takes $\text{im } \beta_{T, \overline{B}}$ to $\text{im } \beta_{T', \overline{B}'}$, hence it takes $\Upsilon_{T, \overline{B}}$ to $\Upsilon_{T', \overline{B}'}$. By the *character lattice of G* we mean the pair $(\Upsilon_{T, \overline{B}}, X(\overline{T}))$ (which is the same, up to an isomorphism, as $(\Upsilon_{T', \overline{B}'}, X(\overline{T}'))$ for any other pair (T', \overline{B}')).

3.7. Let G be a connected reductive k -group, not necessarily split. Consider a split form of G , i.e., a split k -group G_{spl} together with an isomorphism $\theta: \overline{G_{\text{spl}}} \rightarrow \overline{G}$. Choose \overline{T} and \overline{B} as above, then $\overline{T} \subset \overline{B} \subset \overline{G}$. Choose a split maximal torus $T_{\text{spl}} \subset G_{\text{spl}}$ and a Borel subgroup $B_{\text{spl}} \subset G_{\text{spl}}$ containing T_{spl} . We may and shall assume that θ takes $(\overline{T_{\text{spl}}}, \overline{B_{\text{spl}}})$ to $(\overline{T}, \overline{B})$. The isomorphism θ induces an isomorphism

$$\theta_\Xi: \Xi(G_{\text{spl}}) = \Xi(\overline{G_{\text{spl}}}) \rightarrow \Xi(\overline{G})$$

and an isomorphism

$$\theta_{\Xi}^*: \text{Aut } \Xi(\overline{G}) \rightarrow \text{Aut } \Xi(G_{\text{spl}}).$$

From now on and up to the end of Section 3 all our constructions depend on the choice of θ .

We have a finite group $A \subset \text{Aut } \Xi(\overline{G})$, the image of $\text{Gal}(\overline{k}/k)$ under β_{Ξ} . Using θ , we obtain a subgroup $A_{\text{spl}} := \theta_{\Xi}^*(A) \subset \text{Aut } \Xi(G_{\text{spl}})$. We identify $\Xi(G_{\text{spl}}, T_{\text{spl}}, B_{\text{spl}})$ with $\Xi(G_{\text{spl}})$ and denote by

$$A_{\Psi, \text{spl}} \subset \text{Aut } \Xi(G_{\text{spl}}, T_{\text{spl}}, B_{\text{spl}}) \subset \text{Aut } \Psi(G_{\text{spl}}, T_{\text{spl}})$$

the subgroup corresponding to $A_{\text{spl}} \subset \text{Aut } \Xi(G_{\text{spl}})$.

Let $W_{\text{spl}} := W(G_{\text{spl}}, T_{\text{spl}})$ be the Weyl group. Define a subgroup

$$\Upsilon_{\text{spl}} := W_{\text{spl}} \cdot A_{\Psi, \text{spl}} = W_{\text{spl}} \rtimes A_{\Psi, \text{spl}} \subset \text{Aut } \Psi(G_{\text{spl}}, T_{\text{spl}}).$$

The isomorphism θ induces an isomorphism

$$\theta_{\Psi}: \Psi(G_{\text{spl}}, T_{\text{spl}}) = \Psi(\overline{G_{\text{spl}}}, \overline{T_{\text{spl}}}) \rightarrow \Psi(\overline{G}, \overline{T})$$

and an isomorphism of lattices

$$(3.4) \quad \theta_{\Psi}^*: (\Upsilon, \mathbf{X}(\overline{T})) \rightarrow (\Upsilon_{\text{spl}}, \mathbf{X}(T_{\text{spl}})).$$

We have a canonical homomorphism

$$\lambda_{\text{spl}}: \text{Aut}(G_{\text{spl}}) \rightarrow \text{Aut } \Xi(G_{\text{spl}}),$$

see §3.3. We obtain a subgroup

$$(3.5) \quad M_{\text{spl}} = (\lambda_{\text{spl}})^{-1}(A_{\text{spl}}) \subset \text{Aut}(G_{\text{spl}}).$$

If we choose a pinning of $(G_{\text{spl}}, T_{\text{spl}}, B_{\text{spl}})$, then we may write

$$M_{\text{spl}} = \text{Inn}(G_{\text{spl}}) \rtimes A_{G, \text{spl}},$$

where $A_{G, \text{spl}} \subset \text{Aut}(G_{\text{spl}})$ is a suitable subgroup isomorphic to A_{spl} , see §3.3.

3.8. Let G be a reductive k -group. A k -group G' is called a k -form of G if G' becomes isomorphic to G over the algebraic closure \overline{k} of k . Recall that isomorphism classes of k -forms of G are in a natural bijective correspondence with the non-abelian Galois cohomology set $H^1(k, \text{Aut}(G))$. Let $z \in Z^1(k, \text{Aut}(G))$ be a cocycle, then the twisted group ${}_z G$ corresponds to the cohomology class $[z] \in H^1(k, \text{Aut}(G))$. For details on this, see e.g. [Sp2, §§11.3 and 12.3].

Lemma 3.9. *For G and M_{spl} as in 3.7, G is isomorphic to the twisted form ${}_z G_{\text{spl}}$ of G_{spl} for some cocycle $z \in Z^1(k, M_{\text{spl}})$.*

Proof. For $\sigma \in \text{Gal}(\overline{k}/k)$ denote by $\beta(\sigma): \overline{G} \rightarrow \overline{G}$ the corresponding semi-automorphism of \overline{G} and by $\beta_{\text{spl}}(\sigma)$ the corresponding semi-automorphism of $\overline{G_{\text{spl}}}$. Set

$$(3.6) \quad z(\sigma) = \theta^{-1} \circ \beta(\sigma) \circ \theta \circ \beta_{\text{spl}}(\sigma)^{-1}: \overline{G_{\text{spl}}} \rightarrow \overline{G_{\text{spl}}},$$

then an easy calculation shows that $z: \text{Gal}(\bar{k}/k) \rightarrow \text{Aut}(\overline{G_{\text{spl}}})$ is a cocycle, $z \in Z^1(k, \text{Aut}(G_{\text{spl}}))$. We have

$$\theta^*(\beta(\sigma)) := \theta^{-1} \circ \beta(\sigma) \circ \theta = z(\sigma) \beta_{\text{spl}}(\sigma),$$

hence ${}_z G_{\text{spl}}$ is isomorphic to G . We show that $z(\sigma) \in M_{\text{spl}}(\bar{k})$ for all $\sigma \in \text{Gal}(\bar{k}/k)$.

Set

$$z_{\Xi} = \lambda_{\text{spl}} \circ z: \text{Gal}(\bar{k}/k) \xrightarrow{z} \text{Aut}(\overline{G_{\text{spl}}}) \xrightarrow{\lambda_{\text{spl}}} \text{Aut } \Xi(\overline{G_{\text{spl}}}) = \text{Aut } \Xi(G_{\text{spl}}).$$

From (3.6) we have

$$z_{\Xi}(\sigma) = \theta_{\Xi}^{-1} \circ \beta_{\Xi}(\sigma) \circ \theta_{\Xi} \circ \beta_{\text{spl}, \Xi}(\sigma)^{-1}$$

where β_{Ξ} and $\beta_{\text{spl}, \Xi}$ denote the actions of $\text{Gal}(\bar{k}/k)$ on $\Xi(\bar{G})$ and on $\Xi(\overline{G_{\text{spl}}}) = \Xi(G_{\text{spl}})$, respectively. Since G_{spl} is split, the Galois group $\text{Gal}(\bar{k}/k)$ acts trivially on $\Xi(G_{\text{spl}})$, hence $\beta_{\text{spl}, \Xi}(\sigma) = \text{id}$, and we obtain

$$z_{\Xi}(\sigma) = \theta_{\Xi}^{-1} \circ \beta_{\Xi}(\sigma) \circ \theta_{\Xi} = \theta_{\Xi}^*(\beta_{\Xi}(\sigma)).$$

Since by definition $A = \text{im } \beta_{\Xi}$, we have $\beta_{\Xi}(\sigma) \in A$. We see that $z_{\Xi}(\sigma) \in \theta_{\Xi}^*(A) = A_{\text{spl}}$ and

$$z(\sigma) \in \lambda_{\text{spl}}^{-1}(A_{\text{spl}}) = M_{\text{spl}}(\bar{k}).$$

□

4. (G, S) -FIBRATIONS AND (G, S) -VARIETIES

The proof of Theorem 1.3 in the next section relies on the notions of (G, S) -fibration and (G, S) -variety. This section will be devoted to preliminary material on these notions.

4.1. (G, S) -fibrations. Let G be a linear algebraic k -group and S be a k -subgroup. Recall that a (G, S) -fibration is a morphism of k -varieties $\pi: X \rightarrow Y$, where G acts on X on the left, π is constant on G -orbits, and after a surjective étale base change $Y' \rightarrow Y$ there is a G -equivariant isomorphism between $G/S \times_k Y'$ and $X \times_Y Y'$ over Y' , cf. [CKPR, §2.2]. If $S = \{1\}$ then a (G, S) -fibration is the same thing as a left G -torsor. Note that in general, $X \rightarrow Y$ can be both a (G, S_1) -fibration and a (G, S_2) -fibration for non-isomorphic k -subgroups $S_1, S_2 \subset G$. However over an algebraic closure of k , S_1 and S_2 become conjugate.

The following lemma generalizes well-known properties of torsors to the category of (G, S) -fibrations (with the same proofs).

Lemma 4.1. *Let $\pi: X \rightarrow Y$, $\pi_1: X_1 \rightarrow Y_1$ and $\pi_2: X_2 \rightarrow Y_2$ be (G, S) -fibrations.*

- (a) Every G -equivariant morphism $f: X_1 \rightarrow X_2$ is a morphism of (G, S) -fibrations, i.e., gives rise to a Cartesian diagram

$$\begin{array}{ccc} X_1 & \xrightarrow{f} & X_2 \\ \pi_1 \downarrow & & \downarrow \pi_2 \\ Y_1 & \xrightarrow{\bar{f}} & Y_2. \end{array}$$

In other words, $X_1 = X_2 \times_{Y_2} Y_1$, where the G -action on $X_2 \times_{Y_2} Y_1$ is induced by the G -action on X_2 .

- (b) Every G -invariant closed (respectively, open) subvariety $X_0 \subset X$ is of the form $\pi^{-1}(Y_0)$ for some closed (respectively, open) subvariety Y_0 of Y . In particular, X_0 is itself the total space of a (G, S) -fibration $\pi|_{X_0}: X_0 \rightarrow Y_0$.
- (c) The map f in part (a) is dominant if and only if \bar{f} is dominant.

Proof. (a) We first define the map $\bar{f}: Y_1 \rightarrow Y_2$ locally in the étale topology on Y_1 . Let $\{U_\alpha\}$ be an étale open cover of Y_1 such that X_1 is G -equivariantly isomorphic to $G/S \times_k U_\alpha$, over each U_α . Then over each U_α , the map π_1 has a section $s: Y_1 \rightarrow G/S \times_k Y_1$, and we can define \bar{f} by composing s , f and π_2 . The resulting local map is independent of the choice of s ; these maps patch up to a k -morphism $\bar{f}: Y_1 \rightarrow Y_2$ by étale descent.

By the universal property of fibered products there exists a morphism $\phi: X_1 \rightarrow X_2 \times_{Y_2} Y_1$ over Y_1 . This morphism is unique and hence, G -equivariant. We want to show that ϕ is an isomorphism. Note that ϕ is a G -equivariant morphism between (G, S) -fibrations over Y_1 . We want to show that if $Y_1 = Y_2$ and $\bar{f} = \text{id}$ in the above diagram then f is an isomorphism. We do this by constructing f^{-1} . Let $\{U_\alpha\}$ be an étale local cover of Y_1 , trivializing both X_1 and X_2 , that is, over each U_α , X_1 and X_2 are both G -equivariantly isomorphic to $G/S \times_k U_\alpha$. Hence, f^{-1} is (uniquely) defined and is G -equivariant over each U_α . Once again, using étale descent, we see that these local inverses patch together to a well-defined G -equivariant k -morphism $f^{-1}: X_2 \rightarrow X_1$.

(b) Since open subsets are complements of closed subsets, it suffices to consider the case where X_0 is closed. We claim that $\pi(X_0)$ is closed in Y . It is enough to check this claim locally in the étale topology, so we may assume that $X = G/S \times_k Y$ and π is the projection onto the second factor. Since X_0 is G -equivariant, X_0 contains $\{1\} \times_k \pi(X_0)$. Moreover, since X_0 is closed, X_0 contains $\{1\} \times \overline{\pi(X_0)}$. We conclude that $\pi(X_0)$ is contained in $\pi(X_0)$, i.e., $\pi(X_0)$ is closed, as claimed.

After replacing Y by $\pi(X_0)$ and X by $\pi^{-1}(\pi(X_0))$, it now suffices to show that if $X_0 \subset X$ is closed and G -invariant and $\pi(X_0) = Y$ then $X_0 = X$. To do this, we construct the inverse to the inclusion map $X_0 \hookrightarrow X$. We first do this étale-locally, where we may assume $X = G/S \times_k Y$ and hence, $X_0 = X$, then use étale descent to patch together local inverses into a morphism $X \rightarrow X_0$ defined over Y .

(c) By part (b), the closure of $f(X_1)$ in X_2 is of the form $\pi_2^{-1}(C)$ for some closed subset $C \subset Y_2$. Thus f is dominant if and only if $C = Y_2$, that is, if and only if \bar{f} is dominant. \square

Let $N := N_G(S)$ be the normalizer of S in G , $W := N/S$, and $X \rightarrow Y$ be a (G, S) -fibration. Denote the S -fixed point locus in X by X^S . The G -action on X induces an N -action on X^S . Since S acts trivially on X^S , this N -action descends to a W -action on X^S . By trivializing the (G, S) -fibration $X \rightarrow Y$ over an étale cover $Y' \rightarrow Y$, we see that $X^S \rightarrow Y$ is in fact a W -torsor; see [CKPR, Proposition 2.9]. Conversely, starting with a W -torsor $Z \rightarrow Y$, we can build a (G, S) -fibration $X \rightarrow Y$ by setting X to be the “homogeneous fibre space” $G \times^N Z$, i.e., the quotient of $G \times_k Z$ by the left N -action given by $n \cdot (g, x) \rightarrow (gn^{-1}, nx)$. This quotient can either be constructed locally, in the étale topology on Y , by descent, or globally as a geometric quotient in the sense of geometric invariant theory. For details on these constructions, we refer the reader to [CKPR, §2.2].

Proposition 4.2. *Let Var_k be the category of quasi-projective varieties, and $\text{Fib}_{(G,S)}$ be the functor from Var_k to the category of sets which associates to a quasi-projective variety Y the set of isomorphism classes of (G, S) -fibrations over Y , and to a k -morphism of varieties $\tilde{Y} \rightarrow Y$ the pull-back morphism which base-changes (G, S) -fibrations over Y to \tilde{Y} . If $S = \{1\}$, we will write Tor_G in place of $\text{Fib}_{(G,S)}$.*

Then the two constructions described above give rise to an isomorphism between the functors $\text{Fib}_{(G,S)}$ and Tor_W .

Proof. See [CKPR, Proposition 2.10]. \square

4.2. (G, S) -varieties. A k -variety X with a left action of G is called a (G, S) -variety if it contains a dense open subset $X' \subset X$ which is the total space of a (G, S) -fibration $X' \rightarrow Y$.

Lemma 4.3. *Let G be a split reductive k -group, and $T \subset G$ a split k -torus. Let $A \subset \text{Aut } \Xi(G)$ be a finite subgroup, and let $M = \lambda^{-1}(A) \subset \text{Aut}(G)$ be as in § 3.3. Then G and its Lie algebra $\text{Lie}(G)$ are both (M, T^{ad}) -varieties.*

Proof. In the case where $A = \{1\}$, this assertion is proved in [CKPR, Proposition 4.3]. We mimic the argument there for arbitrary finite A . We will only consider the M -action on G ; the case of the M -action on $\text{Lie}(G)$ is similar. We choose (T, B) and a pinning of (G, T, B) as in §§ 3.1 and 3.3. We obtain a subgroup $A_G \subset M$.

Our proof will rely on [CKPR, Proposition 2.16]. To apply this proposition we need to check that the M -action on G is stable, i.e., the M -orbit of $x \in G(\bar{k})$ is closed for x in general position. By [CKPR, Corollary 4.2], the conjugation action of G on itself is stable. Since A is a finite group, the group M contains G^{ad} as a subgroup of finite index, and therefore the M -action on G is also stable.

By [CKPR, Proposition 2.15(i)] we can now conclude that X is an (M, S) -variety for some subgroup $S \subset M$. Moreover, by [CKPR, Proposition 2.16], in order to show that we may take $S = T^{\text{ad}}$, it suffices to exhibit a subset D of $X(k)$ which is dense in G and such that for every $p \in D$ the stabilizer of p in M is conjugate to T^{ad} .

To construct D , let U be a dense open subset of T consisting of points t such that (i) t is regular, and (ii) t is not fixed by any non-trivial element of $\Upsilon = W \rtimes A_\Psi$. Both (i) and (ii) are open conditions (for (i) see [Bo, §12.2]), so U is an open subset of T defined over k . We now claim that the set

$$D := \{gtg^{-1} \mid g \in G(k), t \in U(k)\}$$

has the desired properties.

First, we show that D is dense in G . Indeed, by Chevalley's theorem, cf. [Bo, Theorem 18.2(ii) and Corollary 18.3], $G(k)$ is dense in G , $U(k)$ is dense in T and thus D is dense in the set $\{gtg^{-1} \mid g \in G, t \in U\}$, which is, in turn, dense in G . This shows that D is dense in G .

Secondly, we show that the stabilizer of p in M is conjugate to T^{ad} for every $p \in D$. It suffices to show that for every $t \in U(k)$, the stabilizer of t in M is T^{ad} . Suppose $\alpha \in M = G^{\text{ad}} \rtimes A_G$ stabilizes t . Since t lies in a unique maximal torus of G (see [Bo, Proposition 12.2(4)]), α stabilizes T or equivalently, lies in $L := \text{Stab}_M(T) = N_{G^{\text{ad}}}(T^{\text{ad}}) \rtimes A_G$. The group $\Upsilon = L/T^{\text{ad}}$ acts on T , and by condition (ii) above, no nontrivial element of $\Upsilon = L/T^{\text{ad}} = W \rtimes A_\Psi$ stabilizes t . We see that $\alpha \in T^{\text{ad}}$, as desired. This completes the proof of Lemma 4.3. \square

Proposition 4.4. *Suppose X_1 and X_2 are (G, S) -varieties such that the fixed point loci X_1^S and X_2^S are irreducible. We write $N = N_G(S)$, $W = N/S$.*

Then every W -equivariant dominant rational map $f: X_1^S \dashrightarrow X_2^S$ lifts to a unique G -equivariant dominant rational map $f': X_1 \dashrightarrow X_2$. Moreover, f' is a birational isomorphism if and only if so is f .

Proof. Choose G -invariant dense open subsets $X'_i \subset X_i$ which are total spaces of (G, S) -fibrations, $X'_i \rightarrow Y_i$, for $i = 1, 2$. Since for each $i = 1, 2$, the variety X_i^S is irreducible, the non-empty open subset $(X'_i)^S$ is dense in X_i^S . Hence, the dominant rational map $X_1^S \dashrightarrow X_2^S$ restricts to a dominant rational map $f: (X'_1)^S \dashrightarrow (X'_2)^S$, and we may, without loss of generality, replace X_i by X'_i .

Consider the map $X_i^S \rightarrow Y_i$, it is a W -torsor, see the constructions before Prop. 4.2. The domain of definition of the dominant rational map f is a W -invariant dense open subset of X_1^S . By Lemma 4.1(b) applied to the $(W, \{1\})$ -fibration $X_1^S \rightarrow Y_1$, this open subset is the preimage of a dense open subset in Y_1 . Thus after replacing Y_1 by a dense open subset and X_1^S by its preimage, we may assume that f is a W -equivariant morphism. Now f lifts to a G -equivariant morphism $f': X_1 \rightarrow X_2$ by Proposition 4.2. Lemma 4.1(c) implies that f' is dominant.

To prove uniqueness, assume that f' and $f'': X_1 \dashrightarrow X_2$ are two G -equivariant rational maps lifting f . Then after replacing Y_1 by a suitable G -invariant open subset, and X_1 by its preimage, we may assume that f , f' and f'' are all regular. Hence, $f' = f''$ by Proposition 4.2.

Finally, if f is a birational isomorphism, apply the above procedure to the W -equivariant map f^{-1} to construct a G -equivariant inverse to f' . \square

5. PROOF OF THEOREM 1.3

5.1. Inner and outer forms.

Lemma 5.1. *Let M be a closed algebraic k -subgroup of the group scheme $\mathrm{Aut}(G)$ which contains $\mathrm{Inn}(G)$, and let $z \in Z^1(k, M)$.*

- (a) *If there exists an M -equivariant birational isomorphism $f: G \dashrightarrow \mathrm{Lie}(G)$, then ${}_zG$ is a Cayley group.*
- (b) *If G is Cayley, then any inner form of G is also Cayley.*
- (c) *If G is stably Cayley, then any inner form of G is also stably Cayley.*

Proof. (a) Since f is M -equivariant, we can twist f by z and obtain an ${}_zM$ -equivariant birational isomorphism

$${}_zf: {}_zG \dashrightarrow {}_z\mathrm{Lie}(G).$$

By functoriality of the twisting operation, ${}_z\mathrm{Inn}(G) = \mathrm{Inn}({}_zG) \subset {}_zM$ ([Sp2, Lemma 16.4.6]) and ${}_z\mathrm{Lie}(G) = \mathrm{Lie}({}_zG)$. Thus ${}_zf$ is an ${}_zM$ -equivariant (and, in particular, $\mathrm{Inn}({}_zG)$ -equivariant) rational map ${}_zG \dashrightarrow \mathrm{Lie}({}_zG)$. Twisting f^{-1} by z in a similar manner, we see that ${}_zf$ is, in fact, a birational isomorphism, i.e., a Cayley map for ${}_zG$.

(b) An inner form of G is, by definition, a twisted form ${}_zG$, where $z \in Z^1(k, \mathrm{Inn}(G))$. If G is a Cayley group, then there exists an $\mathrm{Inn}(G)$ -equivariant birational isomorphism $f: G \dashrightarrow \mathrm{Lie}(G)$, hence by (a) ${}_zG$ is a Cayley group.

(c) If G is a stably Cayley group, then $G \times_k \mathbb{G}_m^r$ is Cayley for some r , and we may identify $\mathrm{Inn}(G)$ with $\mathrm{Inn}(G \times_k \mathbb{G}_m^r)$. If $z \in Z^1(k, \mathrm{Inn}(G)) = Z^1(k, \mathrm{Inn}(G \times_k \mathbb{G}_m^r))$, then by (b) the twisted group ${}_z(G \times_k \mathbb{G}_m^r) = {}_zG \times_k \mathbb{G}_m^r$ is Cayley, hence ${}_zG$ is stably Cayley. \square

5.2. The generic torus. Let G be a connected reductive k -group, and let T_{gen} be the generic torus of G , see [V2, §4.2]. It is defined over a field K_{gen} containing k .

Consider the image A of $\mathrm{Gal}(\bar{k}/k)$ in $\mathrm{Aut} \Xi(\bar{G})$, see §3.4. Fix an embedding of \bar{k} into \bar{K}_{gen} , where \bar{K}_{gen} is an algebraic closure of K_{gen} . Then we can identify $\Xi(\bar{G})$ with $\Xi(G_{\bar{K}_{\mathrm{gen}}})$ and regard A as a subgroup of $\mathrm{Aut} \Xi(G_{\bar{K}_{\mathrm{gen}}})$. Denote $\Psi_{\mathrm{gen}}(G) := \Psi(G_{\bar{K}_{\mathrm{gen}}}, (T_{\mathrm{gen}})_{\bar{K}_{\mathrm{gen}}})$. Consider the preimage $\varphi_{\Psi}^{-1}(A) \subset \mathrm{Aut} \Psi_{\mathrm{gen}}(G)$.

Proposition 5.2. *The image \mathfrak{G} of $\mathrm{Gal}(\bar{K}_{\mathrm{gen}}/K_{\mathrm{gen}})$ in $\mathrm{Aut} \Psi_{\mathrm{gen}}(G)$ coincides with $\varphi_{\Psi}^{-1}(A)$.*

Proof. By a theorem of Voskresenskii's [V2, Thm. 4.2.1], the image of the Galois group $\text{Gal}(\overline{K}_{\text{gen}}/\bar{k}K_{\text{gen}})$ in $\text{Aut } \Psi_{\text{gen}}(G)$ coincides with the Weyl group W . It follows that we have a split short exact sequence

$$1 \rightarrow W \rightarrow \mathfrak{G} \rightarrow A_0 \rightarrow 1,$$

where A_0 is the image of $\text{Gal}(\overline{K}_{\text{gen}}/K_{\text{gen}})$ in $\text{Aut } \Xi(G_{\overline{K}_{\text{gen}}})$. Since G splits over \bar{k} , the Galois group $\text{Gal}(\overline{K}_{\text{gen}}/K_{\text{gen}})$ acts on $\Xi(G_{\overline{K}_{\text{gen}}}) = \Xi(\overline{G})$ via $\text{Gal}(\bar{k}/k)$. Since K_{gen} is the function field of a geometrically irreducible variety, we have $\bar{k} \cap K_{\text{gen}} = k$, and by [La, Thm. VI.1.12] the natural homomorphism $\text{Gal}(\overline{K}_{\text{gen}}/K_{\text{gen}}) \rightarrow \text{Gal}(\bar{k}/k)$ is surjective. It follows that $A_0 = A$, which proves the proposition. \square

Corollary 5.3. *The lattice $(\mathfrak{G}, \mathbf{X}((T_{\text{gen}})_{\overline{K}_{\text{gen}}}))$ is isomorphic to the character lattice $\mathbf{X}(G)$ of G , see Definition 3.5.*

5.3. Conclusion of the proof of Theorem 1.3. (a) \implies (b). Let G be a reductive k -group, $f: G \dashrightarrow \text{Lie}(G)$ be a Cayley map, and $T \subset G$ be a maximal k -torus. First note that the indeterminacy locus of f cannot contain T ; otherwise it would contain the union of all gTg^{-1} , which is dense in G , a contradiction. Thus f can be restricted to a rational map $T \dashrightarrow \text{Lie}(G)$. Moreover, since f is G -equivariant and T is its own centralizer, $f(T)$ lies in $\text{Lie}(G)^T = \text{Lie}(T)$. Thus f restricts to a rational map $f|_T: T \dashrightarrow \text{Lie}(T)$ defined over k . Applying the same argument to f^{-1} , we see that $f|_T$ is a birational isomorphism. In particular, since $\text{Lie}(T)$ is a k -vector space, the k -torus T is a k -rational variety.

Finally, replacing k by K and G by $G_K \times_K \mathbb{G}_{m,K}^r$, we conclude that if T is a maximal K -torus of G then $T \times_K \mathbb{G}_{m,K}^r$ is K -rational. Thus T is stably K -rational, as claimed.

(b) \implies (c) is obvious.

(c) \iff (d): By Corollary 5.3, the character lattice $\mathbf{X}(G)$ of G is isomorphic to the character lattice of the generic torus T_{gen} of G . Since a torus T is stably rational if and only if its character lattice $\mathbf{X}(T)$ is quasi-permutation (see [V2, Theorem 4.7.2]), (c) and (d) are equivalent.

(d) \implies (a): Choose $(\overline{T}, \overline{B})$ as in §3.7. Let $(G_{\text{spl}}, T_{\text{spl}}, B_{\text{spl}})$ be a split form with an isomorphism

$$\theta: (\overline{G_{\text{spl}}}, \overline{T_{\text{spl}}}, \overline{B_{\text{spl}}}) \rightarrow (\overline{G}, \overline{T}, \overline{B}).$$

Recall that we have an isomorphism (3.4) between the lattices $\mathbf{X}(G) := (\Upsilon, \mathbf{X}(\overline{T}))$ and $(\Upsilon_{\text{spl}}, \mathbf{X}(T_{\text{spl}}))$. Since by assumption $(\Upsilon, \mathbf{X}(\overline{T}))$ is a quasi-permutation lattice, we see that $(\Upsilon_{\text{spl}}, \mathbf{X}(T_{\text{spl}}))$ is a quasi-permutation lattice as well. The group Υ_{spl} acts on the split torus T_{spl} . Now Lemma 2.3 tells us that, since $(\Upsilon_{\text{spl}}, \mathbf{X}(T_{\text{spl}}))$ is a quasi-permutation lattice, this action is stably linearizable. In particular, the underlying varieties of the torus T_{spl} , viewed with the action of Υ_{spl} , and of its Lie algebra $\text{Lie}(T_{\text{spl}})$, viewed with the natural (faithful) representation of Υ_{spl} , are equivariantly stably birationally

isomorphic. Since T_{spl} and $\text{Lie}(T_{\text{spl}})$ have the same dimension, there exists a Υ_{spl} -equivariant birational isomorphism

$$(5.1) \quad T_{\text{spl}} \times_k \mathbb{G}_m^r \xrightarrow{\sim} \text{Lie}(T_{\text{spl}}) \times_k \mathbb{A}^r,$$

for some integer $r \geq 0$, where Υ_{spl} acts trivially on the split torus \mathbb{G}_m^r and on the affine space \mathbb{A}^r .

Let $G' = G \times_k \mathbb{G}_m^r$, $G'_{\text{spl}} = G_{\text{spl}} \times_k \mathbb{G}_m^r$, $T'_{\text{spl}} = T_{\text{spl}} \times_k \mathbb{G}_m^r$, etc. Then we have canonical isomorphisms $A'_{\text{spl}} \simeq A_{\text{spl}}$, $\Upsilon'_{\text{spl}} \simeq \Upsilon_{\text{spl}}$, and the Υ_{spl} -equivariant birational isomorphism (5.1) gives us a Υ'_{spl} -equivariant isomorphism

$$(5.2) \quad T'_{\text{spl}} \xrightarrow{\sim} \text{Lie}(T'_{\text{spl}}).$$

We will use this birational isomorphism to construct a Cayley map for G' as follows.

We set $M'_{\text{spl}} = (\varphi_{G', \text{spl}})^{-1}(A'_{\text{spl}}) \subset \text{Aut } G'_{\text{spl}}$, see (3.5). Recall that $X_1 := G'_{\text{spl}}$ and $X_2 := \text{Lie}(G'_{\text{spl}})$ are both (M'_{spl}, S) -varieties, with $S := (T'_{\text{spl}})^{\text{ad}}$; see Lemma 4.3. Here $X_1^S = T'_{\text{spl}}$, $X_2^S = \text{Lie}(T'_{\text{spl}})$, and $W := \Upsilon'_{\text{spl}}$. Thus by Proposition 4.4, the Υ'_{spl} -equivariant birational isomorphism (5.2) lifts to an M'_{spl} -equivariant birational isomorphism

$$(5.3) \quad G'_{\text{spl}} \xrightarrow{\sim} \text{Lie}(G'_{\text{spl}}).$$

Finally, recall that G' is a k -form of G'_{spl} . Thus $G' = {}_z G'_{\text{spl}}$ for some $z \in Z^1(k, \text{Aut}(G'_{\text{spl}}))$. Moreover, by Lemma 3.9 we may take $z \in Z^1(k, M'_{\text{spl}}) \subset Z^1(k, \text{Aut}(G'_{\text{spl}}))$. Twisting both sides of (5.3) by z , we obtain a Cayley map for ${}_z G'_{\text{spl}} = G' = G \times_k \mathbb{G}_m^r$; see Lemma 5.1(a). This completes the proof of Theorem 1.3. \square

6. PROOF OF THEOREM 1.4

To show that (a) \implies (b), suppose G is stably Cayley over k . Then $G_{\bar{k}}$ is stably Cayley over \bar{k} , where \bar{k} denotes an algebraic closure of k . By [LPR, Theorem 1.28] $G_{\bar{k}}$ is one of the following groups:

$$\mathbf{SL}_3, \mathbf{SO}_n \ (n \neq 2, 4), \mathbf{Sp}_{2n} \ (n \geq 1), \mathbf{PGL}_n \ (n \geq 2), \mathbf{G}_2.$$

In other words, G is a k -form of one of these groups. (Note that the group \mathbf{SL}_2 , which appears in the statement of [LPR, Theorem 1.28], is isomorphic to \mathbf{Sp}_2 .) If G is an outer form of \mathbf{PGL}_n where $n \geq 4$ is even, then by [CK, Theorem 0.1] the generic torus of G is not stably rational, and by Theorem 1.3 G is not stably Cayley. Thus if G is stably Cayley, then G is one of the groups listed in part (b).

It remains to prove that (b) \implies (a), i.e., that all groups listed in part (b) are stably Cayley.

The classical Cayley map shows that any form of $G := \mathbf{SO}_n$ and \mathbf{Sp}_{2n} is Cayley; see [LPR, Example 1.16]. All forms of the groups \mathbf{SL}_3 and \mathbf{G}_2 are

of rank 2, hence their generic tori are rational by [V2, Example 4.9.7], and by Theorem 1.3 these groups are stably Cayley. Every inner form of \mathbf{PGL}_n is Cayley by [LPR, Example 1.11]. Finally, the generic torus of any form of \mathbf{PGL}_n for n odd is rational, hence stably rational by [VK, Corollary of Theorem 8]. By Theorem 1.3 we conclude that outer forms of \mathbf{PGL}_n for n odd are stably Cayley. This completes the proof of Theorem 1.4. \square

7. STATEMENT OF THEOREM 7.1 AND FIRST REDUCTIONS

In view of Theorem 1.4 it is natural to ask for a classification of stably Cayley semisimple groups, initially over an algebraically closed field of characteristic zero. This problem turns out to be significantly more complicated; a complete solution is out of reach at the moment. Fortunately, for the purpose of proving Theorem 1.5 we can limit our attention to semisimple groups all of whose simple components are of the same type. Theorem 7.1 stated below gives a classification of stably Cayley groups of this form; this theorem will be a key ingredient in our proof of Theorem 1.5 in §16. The proof of Theorem 7.1 will occupy much of the remainder of this paper.

Theorem 7.1. *Suppose H is a simply connected simple group over an algebraically closed field k . Let $G = H^m/C$, where $C \subset H^m$ is a central subgroup. Then G is stably Cayley if and only if G is a direct product $G = G_1 \times_k \cdots \times_k G_s$ of normal subgroups $G_i \subset G$, where each G_i is either a stably Cayley simple group or isomorphic to \mathbf{SO}_4 .*

The “if” direction of Theorem 7.1 is obvious, since the direct product of stably Cayley groups is stably Cayley. Thus we only need to prove the “only if” direction. The proof will proceed by case-by-case analysis, depending on the type of H . We begin with the following easy reduction.

Lemma 7.2. *Let H be a simply connected simple group over an algebraically closed field k and C be a central subgroup of H^m for some $m \geq 1$. Let H_i denote the i^{th} factor of H^m , π_i denote the natural projection $H^m \rightarrow H_i$, and $C_i := \pi_i(C) \subset Z(H_i)$, where $Z(H_i)$ denotes the center of H_i . Assume H^m/C is stably Cayley. Then*

- (a) H_i/C_i is stably Cayley;
- (b) H is of type \mathbf{A}_n ($n \geq 1$), \mathbf{B}_n ($n \geq 2$), \mathbf{C}_n ($n \geq 3$), \mathbf{D}_n ($n \geq 4$), or \mathbf{G}_2 .

Proof. Part (a) is a direct consequence of [LPR, Prop. 4.8]. To prove part (b), note that by [LPR, Thm. 1.28], H_1/C_1 is of one of the types listed in the statement of the lemma. \square

We will now settle two easy cases of Theorem 7.1, where H is of type \mathbf{C}_n ($n \geq 3$) and \mathbf{G}_2 .

Proof of Theorem 7.1 for $H = \mathbf{G}_2$. Here $Z(H) = \{1\}$, so $C \subset Z(H)^m$ is trivial, and

$$H^m/C = H^m = \mathbf{G}_2 \times_k \cdots \times_k \mathbf{G}_2 \text{ (} m \text{ times)}$$

is a product of stably Cayley simple groups. \square

Proof of Theorem 7.1 for H of type \mathbf{C}_n ($n \geq 3$). Let $H = \mathbf{Sp}_{2n}$ and C be a subgroup of $Z(H)^m = \mu_2^m$. We will show that H^m/C is stably Cayley if and only if $C = \{1\}$.

Indeed, if H^m/C is stably Cayley then, by Lemma 7.2, so is H_i/C_i . Here $H_i = \mathbf{Sp}_{2n}$, and C_i is a central subgroup (either μ_2 or $\{1\}$). On the other hand, by [LPR, Theorem 1.28], if the group \mathbf{Sp}_{2n}/C_i for $n \geq 3$ is stably Cayley, then $C_i = \{1\}$. Thus C projects trivially to every H_i , which is only possible if $C = \{1\}$. We conclude that

$$H^m/C = H^m = \mathbf{Sp}_{2n} \times_k \cdots \times \mathbf{Sp}_{2n} \text{ (} m \text{ times)}$$

is a product of Cayley simple groups, as desired. \square

8. QUASI-INVERTIBLE LATTICES

The proof of the “only if” direction of Theorem 7.1 in the remaining cases, where H is of type \mathbf{A}_n , \mathbf{B}_n or \mathbf{D}_n , is more involved. In this section, in preparation for this proof, we will develop a general method for showing that certain lattices are not quasi-permutation. Our main result, Theorem 8.6, goes back to [V1, Thm. 7 and its corollary]; see also [CS1, Proposition 1(ii)] and [CS2, Proposition 9.5(ii)]. For the sake of completeness we supply short proofs for Lemmas 8.4 and 8.5 below.

Rather than working directly with quasi-permutation lattices, it will be convenient to us to also consider their direct summands.

Definition 8.1. A Γ -lattice L is called *quasi-invertible* if it is a direct summand of a quasi-permutation Γ -lattice.

Note that if $L \sim L'$ are equivalent Γ -lattices, then L is a quasi-permutation (or quasi-invertible) if and only if so is L' .

8.2. Let L be a Γ -lattice. We set

$$\text{III}^2(\Gamma, L) = \ker \left[H^2(\Gamma, L) \rightarrow \prod_{\Gamma_c \subset \Gamma} H^2(\Gamma_c, L) \right],$$

where Γ_c runs over the set of all cyclic subgroups of Γ . If L is a quasi-invertible Γ -lattice, then for any subgroup $\Gamma' \subset \Gamma$ we have $\text{III}^2(\Gamma', L) = 0$, cf. [Lo, Prop. 2.9.2(a)]. Note however, that there exist Γ -lattices L such that $\text{III}^2(\Gamma', L) = 0$ for every subgroup Γ' of Γ but L is not quasi-invertible; see the comments at the end of the proof of Prop. 9.9.

8.3. By a *semi-isomorphism* of Γ -lattices L and L' we shall mean a pair of isomorphisms

$$\varphi: \Gamma \xrightarrow{\sim} \Gamma, \quad \psi: L \xrightarrow{\sim} L'$$

such that

$$\psi(\gamma x) = \varphi(\gamma)\psi(x) \text{ for all } \gamma \in \Gamma, x \in L.$$

Suppose L and L' are semi-isomorphic. Then clearly L is permutation (respectively, quasi-permutation, respectively, quasi-invertible) if and only if so is L' .

The following lemmas can be used to show that a given lattice is not quasi-invertible. Let Γ be a finite group. Consider the norm homomorphism

$$N_\Gamma: \mathbb{Z} \rightarrow \mathbb{Z}[\Gamma], \quad N_\Gamma(a) = a \sum_{s \in \Gamma} s \text{ for } a \in \mathbb{Z},$$

and the short exact sequence

$$(8.1) \quad 0 \rightarrow \mathbb{Z} \rightarrow \mathbb{Z}[\Gamma] \rightarrow J_\Gamma \rightarrow 0,$$

where $J_\Gamma = \text{coker } N_\Gamma$.

Lemma 8.4. *Let Γ be a finite group, and $\Gamma' \subset \Gamma$ any subgroup. Then $\text{III}^2(\Gamma', J_\Gamma) \cong H^3(\Gamma', \mathbb{Z})$.*

Proof. From (8.1) we obtain a cohomology exact sequence

$$(8.2) \quad H^2(\Gamma', \mathbb{Z}[\Gamma]) \rightarrow H^2(\Gamma', J_\Gamma) \rightarrow H^3(\Gamma', \mathbb{Z}) \rightarrow H^3(\Gamma', \mathbb{Z}[\Gamma]).$$

We have $H^i(\Gamma', \mathbb{Z}[\Gamma]) = 0$ for $i \geq 1$, hence $H^i(\Gamma', \mathbb{Z}[\Gamma]) = 0$ for $i \geq 1$, and we see from (8.2) that $H^2(\Gamma', J_\Gamma) \cong H^3(\Gamma', \mathbb{Z})$.

Now let $\Gamma_c \subset \Gamma'$ be a *cyclic* subgroup. We have $H^2(\Gamma_c, J_\Gamma) \cong H^3(\Gamma_c, \mathbb{Z})$. By periodicity for cyclic groups, cf. [ANT, IV.8, Thm. 5], we have

$$H^3(\Gamma_c, \mathbb{Z}) \cong H^1(\Gamma_c, \mathbb{Z}) = \text{Hom}(\Gamma_c, \mathbb{Z}) = 0,$$

whence $H^2(\Gamma_c, J_\Gamma) = 0$. Thus $\text{III}^2(\Gamma', J_\Gamma) = H^2(\Gamma', J_\Gamma) \cong H^3(\Gamma', \mathbb{Z})$. \square

Lemma 8.5. *Let $\Gamma = \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$, where p is a prime. Then $H^3(\Gamma, \mathbb{Z}) \cong \mathbb{Z}/p\mathbb{Z}$.*

Proof. For any group Γ , the group $H^3(\Gamma, \mathbb{Z})$ is canonically isomorphic to $H^2(\Gamma, \mathbb{C}^\times)$. The latter group is called the *Schur multiplier* of Γ . For finite abelian groups, the Schur multipliers were computed by Schur in [Sch, §4, VIII]. In particular, by [Sch, §4, VIII] the Schur multiplier of $\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$ is a cyclic group of order p , which proves the lemma.

An alternative proof based on modern references proceeds as follows. For any finite group Γ , the group $H^3(\Gamma, \mathbb{Z})$ is dual to $H^{-3}(\Gamma, \mathbb{Z})$, cf. [CE, Thm. XII.6.6] or [Br, Thm. VI.7.4]. By definition $H^{-3}(\Gamma, \mathbb{Z}) = H_2(\Gamma, \mathbb{Z})$. For an *abelian* group Γ we have $H_2(\Gamma, \mathbb{Z}) = \Lambda^2(\Gamma)$ (the second exterior power of the \mathbb{Z} -module Γ), see [Mi, Thm. 3] or [Br, Thm. V.6.4]. Clearly $\Lambda^2(\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}) \cong \mathbb{Z}/p\mathbb{Z}$, hence $H_2(\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}, \mathbb{Z}) \cong \mathbb{Z}/p\mathbb{Z}$ and $H^3(\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}, \mathbb{Z}) \cong \mathbb{Z}/p\mathbb{Z}$. \square

As an immediate consequence, we obtain the following

Theorem 8.6. *Let $\Gamma = \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$, where p is a prime. Then $\text{III}^2(\Gamma, J_\Gamma) \cong \mathbb{Z}/p\mathbb{Z}$, and therefore the Γ -lattice J_Γ is not quasi-invertible.* \square

Corollary 8.7. *Let Γ be as in Theorem 8.6, and let $\Delta \rightarrow \Gamma$ be a surjective homomorphism of finite groups. Then $\text{III}^2(\Delta, J_\Gamma) \neq 0$, and therefore the Δ -lattice J_Γ is not quasi-invertible.*

Proof. Set $\Delta_1 = \ker[\Delta \rightarrow \Gamma]$. Then Δ_1 acts on J_Γ trivially, hence

$$H^1(\Delta_1, J_\Gamma) = \text{Hom}(\Delta_1, J_\Gamma) = 0.$$

It follows that we have a restriction-inflation exact sequence

$$0 \rightarrow H^2(\Gamma, J_\Gamma) \xrightarrow{\text{Inf}} H^2(\Delta, J_\Gamma) \xrightarrow{\text{Res}} H^2(\Delta_1, J_\Gamma),$$

cf. [ANT, §IV.5, Proposition 5]. In particular, $H^2(\Gamma, J_\Gamma)$ injects into $H^2(\Delta, J_\Gamma)$, hence $\text{III}^2(\Gamma, J_\Gamma) = H^2(\Gamma, J_\Gamma)$ injects into $\text{III}^2(\Delta, J_\Gamma) = H^2(\Delta, J_\Gamma)$. By Theorem 8.6, $\text{III}^2(\Gamma, J_\Gamma) \neq 0$, hence $\text{III}^2(\Delta, J_\Gamma) \neq 0$, and therefore the Δ -lattice J_Γ is not quasi-invertible. \square

9. A FAMILY OF NON-QUASI-INVERTIBLE LATTICES

In this section we create a stock of non-quasi-invertible lattices (i.e., lattices which are not direct summands of quasi-permutation lattices), which will be used to complete the proof of Theorem 7.1.

9.1. Let Δ be a Dynkin diagram, $\Delta = \bigcup_{i=1}^m \Delta_i$, where Δ_i are the connected components of Δ . We assume that each Δ_i is of type **B** or type **D** (note that **B**₁ = **A**₁ and **D**₃ = **A**₃ are allowed). Let S_i be the set of vertices of Δ_i , and $S = \bigcup S_i$ be the set of vertices of Δ .

Consider the vector space V over \mathbb{Q} with basis $(\varepsilon_s)_{s \in S}$. Set $\beta_1 = \frac{1}{2} \sum_{s \in S} \varepsilon_s$. We denote by M the subgroup in V generated by β_1 and by the basis elements ε_s for all $s \in S$. In other words, M is generated by the vectors of the form $\frac{1}{2} \sum_{s \in S} \pm \varepsilon_s$.

Set $W_i = W(\Delta_i)$ and $W = W(\Delta) = \prod_{i=1}^m W_i$. The Weyl group W acts on M .

Proposition 9.2. *Let Δ , S , M , and W be as in §9.1. Assume that $|S| \geq 3$. Then the W -lattice M is not quasi-invertible.*

Remark 9.3. Note that $\text{rank}(M) = \dim(V) = |S|$. If $|S| = 1$ or 2 then M is quasi-permutation by Lemma 2.6.

Proof. First we consider the case $\Delta \cong \mathbf{D}_4$. Then M is not quasi-permutation, see [CK, §7.1]. We will show that M is not quasi-invertible. Indeed, in [CK, §7.1] the authors construct a subgroup $W_2 \subset W$ of order 8, such that M restricted to W_2 is a direct sum of W_2 -sublattices $M = M_1 \oplus M_3$ of ranks 1 and 3, respectively. Now in [Ku, Theorem 1] it is *stated* that the W_2 -lattice M_3 is not quasi-permutation, but it is actually *proved* that $[M_3]^{\text{fl}}$ (see [Lo, §2.7] for the notation) is not invertible. Hence M_3 is not a quasi-invertible W_2 -lattice, and M is not a quasi-invertible W -lattice.

For the rest of the proof we will assume that $\Delta \not\cong \mathbf{D}_4$. Let $\Gamma = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} = \{1, a, b, ab\}$. Then by Theorem 8.6, $\text{III}^2(\Gamma, J_\Gamma) \cong \mathbb{Z}/2\mathbb{Z}$. We shall

embed Γ into W and show that $\text{III}^2(\Gamma, M) = \mathbb{Z}/2\mathbb{Z} \neq 0$, which will imply that M is not quasi-invertible.

Let us denote by S_0 the union of all S_i such that Δ_i is of type **B**. Let I be the set consisting of 0 and of all i such that Δ_i is of type **D**. For each $i \in I$ we will construct three subsets $U_{i,\varkappa} \subset S_i$ ($\varkappa = 1, 2, 3$) satisfying

$$\begin{aligned} U_{i,\varkappa} \cap U_{i,\varkappa'} &= \emptyset \text{ for } \varkappa \neq \varkappa', \\ U_{i,1} \cup U_{i,2} \cup U_{i,3} &= S_i, \\ |U_{i,\varkappa}| &\equiv l_i \pmod{2} \text{ if } \Delta_i \text{ is of type } \mathbf{D}_{l_i}. \end{aligned}$$

Then we will set $U_\varkappa = \bigcup_{i \in I} U_{i,\varkappa}$.

We construct the subsets $U_{0,\varkappa} \subset S_0$ as follows. If $S_0 = \emptyset$, we set $U_{0,\varkappa} = \emptyset$ for all \varkappa . If $S_0 = \{s_{0,1}\}$, we set $U_{0,1} = \{s_{0,1}\}$, $U_{0,2} = U_{0,3} = \emptyset$. If $|S_0| \geq 2$, we set $U_{0,1} = \{s_{0,1}\}$, $U_{0,2} = \{s_{0,2}\}$, $U_{0,3} = S_0 \setminus (U_{0,1} \cup U_{0,2})$. Note that if $|S_0| \geq 3$, then $U_\varkappa \neq \emptyset$ for all \varkappa .

If $\Delta_i \cong \mathbf{D}_l$ with l odd, then $l \geq 3$, and we construct the subsets $U_{i,\varkappa} \subset S_i$ as follows: $U_{i,1} = \{s_{i,1}\}$, $U_{i,2} = \{s_{i,2}\}$, $U_{i,3} = S_i \setminus (U_{i,1} \cup U_{i,2})$. If there exists such i , then $U_\varkappa \neq \emptyset$ for all \varkappa .

If $\Delta_i \cong \mathbf{D}_l$ with l even and $l \neq 4$, then $l \geq 6$, and we construct the subsets $U_{i,\varkappa} \subset S_i$ as follows: $U_{i,1} = \{s_{i,1}, s_{i,2}\}$, $U_{i,2} = \{s_{i,3}, s_{i,4}\}$, $U_{i,3} = S_i \setminus (U_{i,1} \cup U_{i,2})$. Again, if there exists such i , then $U_\varkappa \neq \emptyset$ for all \varkappa .

If $\Delta_i \cong \mathbf{D}_4$, we choose one such $i = i_1$ and construct the subsets $U_{i,\varkappa} \subset S_i$ as follows: $U_{i,1} = \emptyset$, $U_{i,2} = \{s_{i,1}, s_{i,2}\}$, $U_{i,3} = \{s_{i,3}, s_{i,4}\}$. For all the other such i we set $U_{i,1} = S_i$, $U_{i,2} = U_{i,3} = \emptyset$. If there exists such other i , then $U_\varkappa \neq \emptyset$ for all \varkappa . Even if there exists only one such $i = i_1$, but $S_0 \neq \emptyset$, again $U_\varkappa \neq \emptyset$ for all \varkappa .

Recall that $U_\varkappa = \bigcup_{i \in I} U_{i,\varkappa}$. Clearly the subsets $U_\varkappa \subset S$ satisfy

$$\begin{aligned} (9.1) \quad U_\varkappa \cap U_{\varkappa'} &= \emptyset \text{ for } \varkappa \neq \varkappa', \\ (9.2) \quad U_1 \cup U_2 \cup U_3 &= S, \\ (9.3) \quad |U_\varkappa \cap S_i| &\equiv l_i \pmod{2} \text{ if } \Delta_i \text{ is of type } \mathbf{D}_{l_i}. \end{aligned}$$

It is easy to check that, since $\Delta \not\cong \mathbf{D}_4$ and $|S| \geq 3$,

$$(9.4) \quad U_\varkappa \neq \emptyset \text{ for each } \varkappa = 1, 2, 3.$$

For $\gamma \in \Gamma = \{1, a, b, ab\}$ we define subsets $\Xi_\gamma \subset S$ as follows: $\Xi_1 = \emptyset$, $\Xi_a = U_1 \cup U_2$, $\Xi_b = U_2 \cup U_3$, $\Xi_{ab} = U_1 \cup U_3$. Then it follows from (9.3) that

$$(9.5) \quad |\Xi_\gamma \cap S_i| \equiv 0 \pmod{2} \text{ for } \gamma \in \Gamma \text{ if } \Delta_i \text{ is of type } \mathbf{D}_{l_i}.$$

For $s \in S$, we denote by c_s the automorphism of M acting as -1 on ε_s and as 1 on all the other ε_t ($t \in S$, $t \neq s$). Then for $s \in S_0$ we have $c_s \in W$. For $s \in S \setminus S_0$ we have $c_s \notin W$, however, if $s, s' \in S_i$ for some i such that Δ_i is of type **D**, then $c_s c_{s'} \in W_i \subset W$.

For $\gamma \in \Gamma$ we define

$$a(\gamma) = \prod_{s \in \Xi_\gamma} c_s \in \text{Aut}(M).$$

It follows from (9.5) that $a(\gamma) \in W$. Clearly $a(\gamma)^2 = \text{id}$. Moreover, for $\gamma, \gamma' \in \Gamma$ we have

$$(\Xi_\gamma \cup \Xi_{\gamma'}) \setminus (\Xi_\gamma \cap \Xi_{\gamma'}) = \Xi_{\gamma\gamma'},$$

hence $a(\gamma\gamma') = a(\gamma)a(\gamma')$. We see that $a: \Gamma \rightarrow W$ is an injective homomorphism of groups. We identify Γ with $a(\Gamma) \subset W$.

Recall that $\beta_1 = \frac{1}{2} \sum_{s \in S} \varepsilon_s$. For $\gamma \in \Gamma$ we set

$$\beta_\gamma := \gamma \cdot \beta_1 = \frac{1}{2} \left(- \sum_{s \in \Xi_\gamma} \varepsilon_s + \sum_{s \in S \setminus \Xi_\gamma} \varepsilon_s \right).$$

We have

$$\gamma' \cdot \beta_\gamma = \gamma' \gamma \cdot \beta_1 = \beta_{\gamma' \gamma} \text{ for } \gamma', \gamma \in \Gamma.$$

We have

$$(9.6) \quad \beta_1 + \beta_a = \sum_{s \in S \setminus \Xi_a} \varepsilon_s = \sum_{s \in U_3} \varepsilon_s, \quad \beta_1 + \beta_b = \sum_{s \in U_1} \varepsilon_s, \quad \beta_1 + \beta_{ab} = \sum_{s \in U_2} \varepsilon_s.$$

It follows from (9.6) and (9.4) that the vectors $\beta_1 + \beta_a$, $\beta_1 + \beta_b$, and $\beta_1 + \beta_{ab}$ are nonzero, and it follows from (9.1) that these three vectors are linearly independent. We have also

$$\beta_1 + \beta_a + \beta_b + \beta_{ab} = 0.$$

All this means that the Γ -lattice M_0 of rank 3 generated by $\beta_1, \beta_a, \beta_b, \beta_{ab}$ is isomorphic to $J_\Gamma := \mathbb{Z}[\Gamma]/\mathbb{Z}$. By Theorem 8.6, we have $\text{III}^2(\Gamma, M_0) = \mathbb{Z}/2\mathbb{Z}$.

For any \varkappa choose an element $u_\varkappa \in U_\varkappa$ and set $U'_\varkappa = U_\varkappa \setminus \{u_\varkappa\}$. We set $S' = U'_1 \cup U'_2 \cup U'_3$. It follows from (9.6) that the abelian group generated by the sets $\{\varepsilon_s \mid s \in S'\}$ and $\{\beta_1, \beta_a, \beta_b, \beta_{ab}\}$ contains β_1 and all ε_s for $s \in S$, hence it coincides with M . In other words, the set $\{\beta_1, \beta_a, \beta_b\} \cup \{\varepsilon_s \mid s \in S'\}$ of $|S|$ elements generates our lattice M , and hence it is a basis of M . The group Γ acts on ε_s by ± 1 . We see that the Γ -lattice M is a direct sum of M_0 and a number of one-dimensional Γ -lattices. It follows that

$$\text{III}^2(\Gamma, M) = \text{III}^2(\Gamma, M_0) = \mathbb{Z}/2\mathbb{Z},$$

and therefore M is not a quasi-invertible W -lattice. \square

Proposition 9.4. *Let $M = \{(a_1, a_2, a_3) \in \mathbb{Z}^3 \mid a_1 + a_2 + a_3 \equiv 0 \pmod{2}\}$ be the $W := (\mathbb{Z}/2\mathbb{Z})^3$ -lattice with the action of $(\mathbb{Z}/2\mathbb{Z})^3$ on $M \subset \mathbb{Z}^3$ coming from the non-trivial action of $\mathbb{Z}/2\mathbb{Z}$ on \mathbb{Z} . Then M is not quasi-invertible.*

Proof. Let $\varepsilon_1, \varepsilon_2, \varepsilon_3$ be the standard basis of \mathbb{Z}^3 . For $i = 1, 2, 3$ let $c_i \in W$ denote the automorphism of M taking ε_i to $-\varepsilon_i$ and taking each of the other two ε_j to itself. Set $\sigma = c_2 c_3$, $\tau = c_1 c_2$, $\rho = c_1 c_2 c_3$. We consider the following basis of M :

$$e_1 = \varepsilon_2 - \varepsilon_1, \quad e_2 = \varepsilon_2 - \varepsilon_3, \quad e_3 = -\varepsilon_1 - \varepsilon_3.$$

A direct calculation shows that in this new basis $\{e_1, e_2, e_3\}$, the generators σ, τ, ρ of W are given by the following matrices:

$$\sigma = \begin{pmatrix} 0 & 0 & 1 \\ -1 & -1 & -1 \\ 1 & 0 & 0 \end{pmatrix}, \quad \tau = \begin{pmatrix} -1 & -1 & -1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \rho = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

By [Ku, Theorem 1, case W_2], our W -lattice M is not quasi-permutation. Moreover, the pair (W, M) is isomorphic to (W_2, M_3) , where M_3 is the non-quasi-invertible W_2 -lattice mentioned at the beginning of the proof of Proposition 9.2. Therefore, M is not quasi-invertible. \square

9.5. Let $\mathbb{Z}\mathbf{D}_3$ denote the root lattice of \mathbf{D}_3 with basis

$$\varepsilon_1 + \varepsilon_2, \quad \varepsilon_1 - \varepsilon_2, \quad \varepsilon_2 - \varepsilon_3.$$

Let $m \geq 2$. We consider $(\mathbb{Z}\mathbf{D}_3)^m \subset (\mathbb{Q}\mathbf{D}_3)^m$, where $\mathbb{Q}\mathbf{D}_3 = \mathbb{Z}\mathbf{D}_3 \otimes_{\mathbb{Z}} \mathbb{Q}$. Let $L \subset (\mathbb{Q}\mathbf{D}_3)^m$ be the lattice generated by $(\mathbb{Z}\mathbf{D}_3)^m$ and the vector

$$v_1 := \varepsilon_1 + \varepsilon_4 + \varepsilon_7 + \cdots + \varepsilon_{3m-2}.$$

The group $W(\mathbf{D}_3)^m$ acts on L .

Proposition 9.6. *For $m \geq 2$, the $W(\mathbf{D}_3)^m$ -lattice L of §9.5 is not quasi-invertible.*

Proof. We consider the subgroup $\Gamma \subset W(\mathbf{D}_3)^m$ of order 4, generated by two commuting elements of order 2:

$$a = (12) \ c_4 c_5 \ c_7 c_8 \ \dots \ c_{3m-2} c_{3m-1}, \\ b = c_1 c_2 \ (45).$$

Here c_i takes ε_i to $-\varepsilon_i$. Thus $\Gamma = \langle a, b \rangle \subset W(\mathbf{D}_3)^m$. We show that $\text{III}^2(\Gamma, L) = \mathbb{Z}/2\mathbb{Z}$.

Indeed, let $V = (\mathbb{Q}\mathbf{D}_3)^m$ with the basis $\varepsilon_1, \dots, \varepsilon_{3m}$. Let V_0 be the subspace of V generated by

$$\varepsilon_1, \varepsilon_2, \ \varepsilon_4, \varepsilon_5, \ \dots, \varepsilon_{3m-2}, \varepsilon_{3m-1}.$$

It is Γ -invariant. Set $L_0 = L \cap V_0$. Clearly L/L_0 injects into V/V_0 . Since Γ acts trivially on V/V_0 , we see that $L/L_0 \cong \mathbb{Z}^m$ with trivial Γ -action. Thus we have a short exact sequence of Γ -lattices

$$0 \rightarrow L_0 \rightarrow L \rightarrow \mathbb{Z}^m \rightarrow 0.$$

Since \mathbb{Z}^m is a permutation Γ -lattice, we see that

$$\text{III}^2(\Gamma, L) \cong \text{III}^2(\Gamma, L_0).$$

We prove that $\text{III}^2(\Gamma, L_0) = \mathbb{Z}/2\mathbb{Z}$.

For $s \in \Gamma$ we set $v_s = s \cdot v_1$. If $m > 2$ we set

$$\delta = \varepsilon_7 + \varepsilon_{10} + \cdots + \varepsilon_{3m-2}.$$

If $m = 2$ we set $\delta = 0$. We obtain

$$\begin{aligned} v_1 &= \varepsilon_1 + \varepsilon_4 + \delta, \\ v_a &= \varepsilon_2 - \varepsilon_4 - \delta, \\ v_b &= -\varepsilon_1 + \varepsilon_5 + \delta, \\ v_{ab} &= -\varepsilon_2 - \varepsilon_5 - \delta. \end{aligned}$$

Clearly

$$v_1 + v_a + v_b + v_{ab} = 0.$$

Set $U = \langle v_1, v_a, v_b, v_{ab} \rangle$, then $U \cong J_\Gamma := \mathbb{Z}[\Gamma]/\mathbb{Z}$, and by Theorem 8.6 we have $\text{III}^2(\Gamma, U) = \mathbb{Z}/2\mathbb{Z}$.

Set $\beta_1 = v_1$, $\beta_2 = v_a$, $\beta_3 = v_b$. We set

$$\begin{aligned} \beta_4 &= \varepsilon_4 - \varepsilon_5, \\ \beta_5 &= \varepsilon_7 + \varepsilon_8, \\ \beta_6 &= \varepsilon_7 - \varepsilon_8, \\ &\dots\dots\dots \\ \beta_{2m-1} &= \varepsilon_{3m-2} + \varepsilon_{3m-1}, \\ \beta_{2m} &= \varepsilon_{3m-2} - \varepsilon_{3m-1}. \end{aligned}$$

By Lemma 9.7 below, the set $\beta := \{\beta_1, \dots, \beta_{2m}\}$ is a basis of L_0 . We have $U = \langle \beta_1, \beta_2, \beta_3 \rangle$. Our Γ -lattice L_0 decomposes into a direct sum of Γ -sublattices

$$L_0 = U \oplus \langle \beta_4 \rangle \oplus \dots \oplus \langle \beta_{2m} \rangle.$$

For $4 \leq i \leq 2m$ the Γ -lattice $\langle \beta_i \rangle$ is of rank 1, hence quasi-permutation, and therefore $\text{III}^2(\Gamma, \langle \beta_i \rangle) = 0$. It follows that $\text{III}^2(\Gamma, L_0) = \text{III}^2(\Gamma, U) = \mathbb{Z}/2\mathbb{Z}$, hence $\text{III}^2(\Gamma, L) = \mathbb{Z}/2\mathbb{Z}$. Thus L is not quasi-invertible. \square

Lemma 9.7. *The set $\beta := \{\beta_1, \dots, \beta_{2m}\}$ is a basis of L_0 .*

Proof. First note that $\beta \subset L_0$. Since the set β has $2m$ elements and the lattice L_0 is of rank $2m$, it suffices to show that β generates L_0 .

Recall that $L_0 = L \cap V_0$ and that L is generated by $(\mathbb{Z}\mathbf{D}_3)^m$ and v_1 . Since $v_1 \in V_0$, we see that L_0 is generated by v_1 and $(\mathbb{Z}\mathbf{D}_3)^m \cap V_0$. Since $v_1 = \beta_1 \in \beta$, it suffices to prove that $(\mathbb{Z}\mathbf{D}_3)^m \cap V_0 \subset \langle \beta \rangle$. Clearly $(\mathbb{Z}\mathbf{D}_3)^m \cap V_0$ is generated by the vectors

$$\varepsilon_1 + \varepsilon_2, \varepsilon_1 - \varepsilon_2, \varepsilon_4 + \varepsilon_5, \varepsilon_4 - \varepsilon_5, \dots, \varepsilon_{3m-2} + \varepsilon_{3m-1}, \varepsilon_{3m-2} - \varepsilon_{3m-1}.$$

Note that all the vectors in this list starting $\varepsilon_4 - \varepsilon_5$ are contained in β . It remains to show that the vectors $\varepsilon_1 + \varepsilon_2, \varepsilon_1 - \varepsilon_2, \varepsilon_4 + \varepsilon_5$ are contained in $\langle \beta \rangle$.

Note that $2\delta \in \langle \beta \rangle$ (because $2\varepsilon_7 \in \langle \beta \rangle, \dots, 2\varepsilon_{3m-2} \in \langle \beta \rangle$). We have

$$\beta_1 + \beta_2 = v_1 + v_a = \varepsilon_1 + \varepsilon_2,$$

hence $\varepsilon_1 + \varepsilon_2 \in \langle \beta \rangle$. We have

$$\beta_1 + \beta_3 = v_1 + v_b = \varepsilon_4 + \varepsilon_5 + 2\delta,$$

hence $\varepsilon_4 + \varepsilon_5 \in \langle \beta \rangle$. Since also $\varepsilon_4 - \varepsilon_5 \in \beta \subset \langle \beta \rangle$, we see that $2\varepsilon_4 \in \langle \beta \rangle$. We have

$$\beta_1 - \beta_2 = v_1 - v_a = \varepsilon_1 - \varepsilon_2 + 2\varepsilon_4 + 2\delta,$$

hence $\varepsilon_1 - \varepsilon_2 \in \langle \beta \rangle$. We conclude that $(\mathbb{Z}\mathbf{D}_3)^m \cap V_0 \subset \langle \beta \rangle$, hence β generates L_0 and is a basis of L_0 . This completes the proofs of Lemma 9.7 and of Proposition 9.6. \square

9.8. Let $\mathbb{Z}\mathbf{A}_{n-1}$ denote the root lattice of \mathbf{A}_{n-1} , and Λ_n denote the weight lattice of \mathbf{A}_{n-1} . Let $m \geq 2$. We consider $(\mathbb{Z}\mathbf{A}_2)^m \subset (\Lambda_3)^m$. Let $(\mathbb{Z}\mathbf{A}_2)^{(i)} \subset \Lambda_3^{(i)}$ be the i^{th} factor. Let $\omega_1^{(i)}, \omega_2^{(i)}$ be the basis of $\Lambda_3^{(i)}$ consisting of fundamental weights.

Let $\mathbf{a} = (a_1, \dots, a_m)$, where each a_i equals 1 or 2. In particular, let $\mathbf{1}_m = (1, \dots, 1)$. Let $L_{\mathbf{a}}$ denote the $(S_3)^m$ -lattice generated by $(\mathbb{Z}\mathbf{A}_2)^m$ and the vector

$$x_{\mathbf{a}} := \sum_{i=1}^m a_i \omega_1^{(i)}.$$

Proposition 9.9. *For $m \geq 2$ and for any \mathbf{a} as in §9.8, (i.e., each a_i equals 1 or 2), the $(S_3)^m$ -lattice $L_{\mathbf{a}}$ of §9.8 is not quasi-invertible.*

Proof. First we note that $L_{\mathbf{a}}$ is semi-isomorphic (see §8.3) to $L_{\mathbf{1}_m}$ with respect to some automorphism of $(S_3)^m$. Indeed, let α_1, α_2 be the standard basis of $R = \mathbf{A}_2$ (and of $\mathbb{Z}\mathbf{A}_2$). Let

$$\omega_1 = \frac{1}{3}(2\alpha_1 + \alpha_2), \quad \omega_2 = \frac{1}{3}(\alpha_1 + 2\alpha_2)$$

be the fundamental weights, this is the standard basis of Λ_3 . Let $\bar{\omega}_1, \bar{\omega}_2$ be their images in $\Lambda_3/\mathbb{Z}\mathbf{A}_2 \cong \mathbb{Z}/3\mathbb{Z}$ (this isomorphism is not canonical). Since

$$\omega_1 + \omega_2 = \alpha_1 + \alpha_2 \in \mathbb{Z}\mathbf{A}_2,$$

we have $\bar{\omega}_1 + \bar{\omega}_2 = 0$, hence $\bar{\omega}_2 = 2\bar{\omega}_1$. Thus the nontrivial automorphism σ of the Dynkin diagram \mathbf{A}_2 , acting on $\Lambda_3/\mathbb{Z}\mathbf{A}_2$, takes $\bar{\omega}_1$ to $\bar{\omega}_2 = 2\bar{\omega}_1$.

Now let \mathbf{a} be as in §9.8. Write $D = (\mathbf{A}_2)^m$, $D = D_1 \cup \dots \cup D_m$. For each $i = 1, \dots, m$ we define an automorphism τ_i of $D_i = \mathbf{A}_2$. If $a_i = 1$, we set $\tau_i = \text{id}$, while if $a_i = 2$, we set $\tau_i = \sigma_i$, where σ_i is the nontrivial automorphism of D_i . Then the automorphism $\tau = \prod_i \tau_i$ of $D = (\mathbf{A}_2)^m$ acts on $(\Lambda_3)^m$ and takes $L_{\mathbf{1}_m}$ to $L_{\mathbf{a}}$. We see that the $(S_3)^m$ -lattices $L_{\mathbf{1}_m}$ and $L_{\mathbf{a}}$ are semi-isomorphic with respect to the induced automorphism τ_* of $(S_3)^m$. Thus, in order to prove that the $(S_3)^m$ -lattice $L_{\mathbf{a}}$ is not quasi-invertible, it suffices to show that $L_{\mathbf{1}_m}$ is not quasi-invertible.

Let $\alpha_1^{(i)}, \alpha_2^{(i)}$ be the standard basis of $(\mathbb{Z}\mathbf{A}_2)^{(i)}$. Let $\omega_1^{(i)}, \omega_2^{(i)}$ be the standard basis of $\Lambda_3^{(i)}$. Then

$$\omega_1^{(i)} = \frac{1}{3}(2\alpha_1^{(i)} + \alpha_2^{(i)}).$$

Let $\alpha_1, \dots, \alpha_{3m-1}$ be the standard basis of $\mathbb{Z}\mathbf{A}_{3m-1}$. Let $\lambda_1, \dots, \lambda_{3m-1}$ be the standard basis of Λ_{3m} . Then

$$(9.7) \quad \lambda_1 = \frac{1}{3m}((3m-1)\alpha_1 + (3m-2)\alpha_2 + \dots + 2\alpha_{3m-2} + \alpha_{3m-1}).$$

We embed $(\mathbb{Z}\mathbf{A}_2)^m$ into $\mathbb{Z}\mathbf{A}_{3m-1}$ as follows:

$$\alpha_1^{(i)} \mapsto \alpha_{3(i-1)+1}, \quad \alpha_2^{(i)} \mapsto \alpha_{3(i-1)+2}$$

(i.e., $\alpha_1^{(1)} \mapsto \alpha_1$, $\alpha_2^{(1)} \mapsto \alpha_2$, $\alpha_1^{(2)} \mapsto \alpha_4$, $\alpha_2^{(2)} \mapsto \alpha_5$, etc.). This embedding induces an embedding

$$\psi: (\mathbb{Q}\mathbf{A}_2)^m \hookrightarrow \mathbb{Q}\mathbf{A}_{3m-1}.$$

Set

$$M = \Lambda_{3m} \cap \psi((\mathbb{Q}\mathbf{A}_2)^m).$$

We check that $M = \psi(L_{\mathbf{1}_m})$. Indeed, Λ_{3m} is generated by $\mathbb{Z}\mathbf{A}_{3m-1}$ and λ_1 , hence $\{\alpha_1, \dots, \alpha_{3m-1}, \lambda_1\}$ is a generating set for Λ_{3m} . From (9.7) we see that

$$\alpha_{3m-1} = 3m\lambda_1 - (3m-1)\alpha_1 - (3m-2)\alpha_2 - \dots - 2\alpha_{3m-2},$$

hence the set $\{\alpha_1, \dots, \alpha_{3m-2}, \lambda_1\}$ is a basis for Λ_{3m} . The set

$$\Xi := \{\alpha_1, \alpha_2, \alpha_4, \alpha_5, \dots, \alpha_{3(m-2)+1}, \alpha_{3(m-2)+2}, \alpha_{3(m-1)+1}, \alpha_{3(m-1)+2}\}$$

is contained in M . It is easy to see that Ξ together with

$$\begin{aligned} \mu &:= m\lambda_1 - (m-1)\alpha_3 - (m-2)\alpha_6 - \dots - \alpha_{3m-3} \\ &= \frac{1}{3}((3m-1)\alpha_1 + (3m-2)\alpha_2 + (3m-4)\alpha_4 + (3m-5)\alpha_5 + \dots + 2\alpha_{3m-2} + \alpha_{3m-1}) \end{aligned}$$

is a basis for M . Now

$$\begin{aligned} \mu &- (m-1)(\alpha_1 + \alpha_2) - (m-2)(\alpha_4 + \alpha_5) - \dots - 1(\alpha_{3(m-2)+1} + \alpha_{3(m-2)+2}) \\ &= \frac{1}{3}((2\alpha_1 + \alpha_2) + (2\alpha_4 + \alpha_5) + \dots + (2\alpha_{3(m-1)+1} + \alpha_{3(m-1)+2})) \\ &= \psi(\omega_1^{(1)} + \omega_1^{(2)} + \dots + \omega_1^{(m)}). \end{aligned}$$

We see that M is generated by $\psi((\mathbb{Z}\mathbf{A}_2)^m)$ and $\psi(\omega_1^{(1)} + \omega_1^{(2)} + \dots + \omega_1^{(m)})$, hence $M = \psi(L_{\mathbf{1}_m})$, thus M is isomorphic to $L_{\mathbf{1}_m}$. Therefore, it suffices to prove that M is not quasi-invertible.

On the other hand, Λ_{3m}/M injects into the \mathbb{Q} -vector space $\mathbb{Q}\mathbf{A}_{3m-1}/\psi((\mathbb{Q}\mathbf{A}_2)^m)$ with basis $\overline{\alpha_3}, \overline{\alpha_6}, \dots, \overline{\alpha_{3(m-1)}}$ on which $(S_3)^m$ acts trivially. Thus we obtain a short exact sequence

$$0 \rightarrow M \rightarrow \Lambda_{3m} \rightarrow \mathbb{Z}^{m-1} \rightarrow 0,$$

where \mathbb{Z}^{m-1} is a trivial, hence permutation, $(S_3)^m$ -lattice. It follows that the $(S_3)^m$ -lattices M and Λ_{3m} are equivalent, and therefore it suffices to show that Λ_{3m} is not a quasi-invertible $(S_3)^m$ -lattice.

Now we embed $S_3 \times S_3$ into $(S_3)^m$ as follows: $(s, t) \in S_3 \times S_3$ maps to $(s, t, \dots, t) \in (S_3)^m$. With the notation of [LPR, (6.4)] we have $\Lambda_{3m} =$

$Q_{3m}(1)$. By [LPR, Proposition 7.1], with respect to the above embedding $S_3 \times S_3 \hookrightarrow (S_3)^m$, we have

$$Q_{3m}(1)|_{S_3 \times S_3} \sim \Lambda_6|_{S_3 \times S_3}.$$

By [LPR, Proposition 7.4(b)], Λ_6 is not a quasi-permutation $S_3 \times S_3$ -lattice, and it is actually proved there that Λ_6 is not a quasi-invertible $S_3 \times S_3$ -lattice although $\text{III}^2(\Gamma', \Lambda_6) = 0$ for every subgroup Γ' of $S_3 \times S_3$. Thus Λ_{3m} is not a quasi-invertible $S_3 \times S_3$ -lattice, hence it is not a quasi-invertible $(S_3)^m$ -lattice. Thus L_{1_m} is not a quasi-invertible $(S_3)^m$ -lattice, and therefore $L_{\mathbf{a}}$ is not a quasi-invertible $(S_3)^m$ -lattice for any \mathbf{a} as in §9.8. This completes the proof of Proposition 9.9. \square

10. STANDARD SUBGROUPS

10.1. Let e_1, \dots, e_m be the standard $\mathbb{Z}/n\mathbb{Z}$ -basis of $(\mathbb{Z}/n\mathbb{Z})^m$. We say that a subgroup $S \subset (\mathbb{Z}/n\mathbb{Z})^m$ is *standard* if S is generated by $n_1 e_1, \dots, n_r e_r$ for some $1 \leq r \leq m$ and some integers n_1, \dots, n_r , where n_i divides n_{i+1} for $i = 1, \dots, r-1$.

Let W be a finite group, P be a W -lattice, and $\lambda: P \rightarrow \mathbb{Z}/n\mathbb{Z}$ be a surjective morphism of W -modules, where W acts trivially on $\mathbb{Z}/n\mathbb{Z}$. Given a subgroup S of $(\mathbb{Z}/n\mathbb{Z})^m$, let P_S^m denote the preimage of S in P^m with respect to the homomorphism $\lambda^m: P^m \rightarrow (\mathbb{Z}/n\mathbb{Z})^m$. We regard P_S^m as a W -submodule of P^m , where W acts diagonally on P^m .

Lemma 10.2. *Let W , P , n and λ be as in §10.1. For every subgroup $S \subset (\mathbb{Z}/n\mathbb{Z})^m$ there exists a standard subgroup $S_{\text{st}} \subset (\mathbb{Z}/n\mathbb{Z})^m$ with the following property: there exist an isomorphism $g_P: P_S^m \xrightarrow{\sim} P_{S_{\text{st}}}^m$ of W -modules and an automorphism g of $(\mathbb{Z}/n\mathbb{Z})^m$ taking S to S_{st} such that the following diagram commutes:*

$$\begin{array}{ccccc} P_S^m & \xrightarrow{\lambda^m} & S \hookrightarrow & (\mathbb{Z}/n\mathbb{Z})^m \\ \downarrow g_P & & \downarrow & \downarrow g \\ P_{S_{\text{st}}}^m & \xrightarrow{\lambda^m} & S_{\text{st}} \hookrightarrow & (\mathbb{Z}/n\mathbb{Z})^m. \end{array}$$

Proof. The homomorphism $\lambda^m: P^m \rightarrow (\mathbb{Z}/n\mathbb{Z})^m$ can be written as

$$\lambda^m = \lambda \otimes_{\mathbb{Z}} \text{id}: P \otimes_{\mathbb{Z}} \mathbb{Z}^m \longrightarrow \mathbb{Z}/n\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}^m.$$

Since the diagram

$$\begin{array}{ccc} P \otimes_{\mathbb{Z}} \mathbb{Z}^m & \xrightarrow{\lambda \otimes \text{id}_{\mathbb{Z}^m}} & \mathbb{Z}/n\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}^m \\ \downarrow \text{id}_P \otimes g & & \downarrow \text{id}_{\mathbb{Z}/n\mathbb{Z}} \otimes g \\ P \otimes_{\mathbb{Z}} \mathbb{Z}^m & \xrightarrow{\lambda \otimes \text{id}_{\mathbb{Z}^m}} & \mathbb{Z}/n\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}^m \end{array}$$

commutes, we see that the group $\text{GL}_m(\mathbb{Z}) = \text{Aut}(\mathbb{Z}^m)$ acts compatibly on the source and the target of the homomorphism $\lambda^m = \lambda \otimes_{\mathbb{Z}} \mathbb{Z}^m$. It suffices to

show that for every subgroup $S \subset (\mathbb{Z}/n\mathbb{Z})^m$ there exists $g \in \mathrm{GL}_m(\mathbb{Z})$ such that $g(S)$ is standard.

Let $\pi: \mathbb{Z}^m \rightarrow (\mathbb{Z}/n\mathbb{Z})^m$ be the natural projection. Then $\pi^{-1}(S)$ is a finite index subgroup of \mathbb{Z}^m . There exist a basis b_1, \dots, b_m of \mathbb{Z}^m and integers $n_1 | n_2 | \dots | n_m$, such that $n_1 b_1, \dots, n_m b_m$ form a basis of $\pi^{-1}(S)$; cf. [La, Theorem III.7.8]. Now let $g \in \mathrm{GL}_m(\mathbb{Z})$ be the element that takes the basis b_1, \dots, b_m to the standard basis of \mathbb{Z}^m . Then $g(\pi^{-1}(S))$ is the subgroup $n_1 \mathbb{Z} \times \dots \times n_m \mathbb{Z}$ of \mathbb{Z}^m and thus $S_{\mathrm{st}} := g(S) = \langle n_1 e_1, \dots, n_m e_m \rangle = \langle n_1 e_1, \dots, n_r e_r \rangle$ is standard, where $r \leq m$ is the largest integer such that n does not divide n_r . \square

Set $Q = \ker \lambda \subset P$. For a subgroup $S_1 \subset \mathbb{Z}/n\mathbb{Z}$ we set $P_{S_1}^1 = \lambda^{-1}(S_1)$, then $Q \subset P_{S_1}^1 \subset P$.

Corollary 10.3. *With the notation of §10.1 assume that S is cyclic. Then*

$$P_S^m \cong P_{S_1}^1 \oplus Q^{m-1}$$

for some subgroup $S_1 \subset \mathbb{Z}/n\mathbb{Z}$ isomorphic to S .

Proof. By Lemma 10.2, we have $P_S^m \cong P_{S_{\mathrm{st}}}^m$. Since S is cyclic, say of order s , the group S_{st} is generated by $(n/s)e_1$. Set $S_1 = \langle (n/s)e_1 \rangle \subset \mathbb{Z}/n\mathbb{Z}$, then clearly

$$P_{S_{\mathrm{st}}}^m = P_{S_1}^1 \oplus Q^{m-1},$$

and the corollary follows. \square

Corollary 10.4. *With the notation of §10.1 assume that S contains an element of order n . Then P_S^m has a direct summand isomorphic to P .*

Proof. By Lemma 10.2, P_S^m is isomorphic to $P_{S_{\mathrm{st}}}^m$ for some standard subgroup $S_{\mathrm{st}} \subset (\mathbb{Z}/n\mathbb{Z})^m$. From the definition of a standard subgroup we see that

$$P_{S_{\mathrm{st}}}^m = P_{S_1}^1 \oplus \dots \oplus P_{S_m}^1,$$

where $S_i \subset \mathbb{Z}/n\mathbb{Z}$ is generated by $n_i e_i$ (for $i > r$ we take $n_i = 0$). Since S_{st} contains an element of order n , we see that $n_1 = 1$, hence S_1 is generated by e_1 , i.e., $S_1 = \mathbb{Z}/n\mathbb{Z}$ and $P_{S_1}^1 = P$. Thus P_S^m has a direct summand isomorphic to P . \square

11. COORDINATE AND ALMOST COORDINATE SUBSPACES

In this section we will collect several elementary results from combinatorial linear algebra, which will be needed to complete the proof of Theorem 7.1.

Let F be a field, F^m be an m -dimensional F -vector space equipped with the standard basis $e_1 = (1, 0, \dots, 0), \dots, e_m = (0, \dots, 0, 1)$.

Recall that the *Hamming weight* of a vector $v = (a_1, \dots, a_m) \in F^m$ is defined as the number of non-zero elements among a_1, \dots, a_m . We will say $v \in F^m$ is *defective* if its Hamming weight is $< m$ or, equivalently, if at least one of its coordinates is 0. The following lemma is well known; a variant of it

is used to construct the standard open cover of the Grassmannian $\text{Gr}(m, d)$ by $d(m - d)$ -dimensional affine spaces, see, e.g., [GH, §1.5]. For the sake of completeness, we supply a short proof.

Lemma 11.1. *Let V be a vector subspace of F^m of dimension $d \geq 2$. Then V has a basis consisting of defective vectors.*

Proof. Let A be an $m \times d$ matrix whose columns form a basis of V . Then

$$V = \{Aw \mid w \in F^d\}.$$

Note that for any invertible $d \times d$ matrix B , the columns of AB will also form a basis of V . Since the columns of A are linearly independent, A has a nondegenerate $d \times d$ submatrix M . Let $B = M^{-1}$. Then the $m \times d$ matrix AB has a $d \times d$ identity submatrix. Since $d \geq 2$, this implies that every column of AB is defective. The columns of AB thus give us a desired basis of defective vectors for V . \square

Definition 11.2. We will say that V is a *coordinate subspace* if V has a basis of coordinate vectors e_{i_1}, \dots, e_{i_d} , for some $I = \{i_1, \dots, i_d\} \subset \{1, \dots, m\}$. We will denote such a subspace by F_I .

Lemma 11.3. *Let $V \subset F^m$ be an F -subspace. Suppose $V \cap F_I$ is coordinate for every $I \subsetneq \{1, \dots, m\}$, then either*

- *V is the 1-dimensional subspace spanned by a vector $\mathbf{a} = (a_1, \dots, a_m)$, where $a_1 \neq 0, \dots, a_m \neq 0$, or*
- *V is coordinate.*

Proof. Assume that V is not of the form $\text{Span}_F(a_1, \dots, a_m)$, where $a_1 \neq 0, \dots, a_m \neq 0$. Then V has a basis of defective vectors. Indeed, if $\dim(V) = 1$ this is obvious, since every vector in V is defective. The case where $\dim(V) \geq 2$ is covered by Lemma 11.1.

Clearly $v \in F^m$ is defective if and only if $v \in F_I$ for some $I \subsetneq \{1, \dots, m\}$. Thus V is spanned by $V \cap F_I$, as I ranges over the proper subsets of $\{1, \dots, m\}$. By our assumption, each $V \cap F_I$ is coordinate and therefore is spanned by coordinate vectors. We conclude that V itself is spanned by coordinate vectors, i.e., is coordinate, as desired. \square

Definition 11.4. We will say that $V \subset F^m$ is *almost coordinate* if V has a basis of the form

$$(11.1) \quad e_{i_1}, \dots, e_{i_r}, e_{j_1} + e_{h_1}, \dots, e_{j_s} + e_{h_s},$$

where $i_1, \dots, i_r, j_1, \dots, j_s, h_1, \dots, h_s$ are distinct integers between 1 and m . We will refer to such a basis as an *almost coordinate basis* of V .

Remark 11.5. An almost coordinate subspace $V \subset F^m$ has a unique almost coordinate basis. In other words, the set of integers $\{i_1, \dots, i_r\}$ and the set of unordered pairs $\{\{j_1, h_1\}, \dots, \{j_s, h_s\}\}$ in (11.1) are uniquely determined by V .

Indeed, $\{i_1, \dots, i_r\}$ is the set of subscripts $i \in \{1, \dots, m\}$ such that the coordinate vector e_i lies in V . The set $\{\{j_1, h_1\}, \{j_2, h_2\}, \dots, \{j_s, h_s\}\}$ is then the set of unordered pairs $\{j, h\}$ such that $j, h \notin \{i_1, \dots, i_r\}$ and $e_j + e_h \in V$.

Proposition 11.6. *Let $F = \mathbb{Z}/2\mathbb{Z}$, and let $V \subset F^m$ be an F -subspace for some $m \geq 4$. Assume $V \cap F_I$ is almost coordinate in $F_I \cong (\mathbb{Z}/2\mathbb{Z})^r$ for every $I = \{i_1, \dots, i_r\} \subsetneq \{1, \dots, m\}$. Then either*

- *V is the 1-dimensional subspace spanned by $(1, \dots, 1)$, or*
- *V is almost coordinate.*

Proof. Assume that V is not of the form $\text{Span}_F(1, \dots, 1)$. Then, once again, Lemma 11.1 tells us that V has a basis of defective vectors, i.e., V is spanned by $V \cap F_I$, as I ranges over the proper subsets of $\{1, \dots, m\}$. By our assumption, each $V \cap F_I$ is almost coordinate and therefore is spanned by vectors of Hamming weight 1 or 2. We conclude that V itself is spanned by vectors of weight 1 or 2. Choose a spanning set of the form

$$(11.2) \quad e_{i_1}, \dots, e_{i_r}, e_{j_1} + e_{h_1}, \dots, e_{j_s} + e_{h_s}$$

of minimal total Hamming weight, i.e., with minimal value of $r + 2s$. Here

$$i_1, \dots, i_r, j_1, h_1, \dots, j_s, h_s \in \{1, \dots, m\}$$

and $j_1 \neq h_1, \dots, j_s \neq h_s$. We claim that (11.2) is an almost coordinate basis of V , i.e., that the subscripts

$$(11.3) \quad i_1, \dots, i_r, j_1, \dots, j_s, h_1, \dots, h_s$$

are all distinct. Clearly, Proposition 11.6 follows from this claim.

It thus remains to prove the claim. The minimality of the total Hamming weight of our spanning set (11.2) implies that we cannot remove any vectors, i.e., that it is a basis of V . In particular, the subscripts i_1, \dots, i_r and the pairs $(j_1, h_1), \dots, (j_s, h_s)$ are distinct. If there is an overlap among the subscripts (11.3), then, after permuting coordinates, we have either $i_1 = j_1$ or $j_1 = j_2$. We will now show that neither of these equalities can occur.

If $i_1 = j_1$ then we may replace $e_{j_1} + e_{h_1}$ by

$$e_{h_1} = (e_{j_1} + e_{h_1}) - e_{i_1} \in V.$$

We will obtain a new spanning set consisting of vectors of weight 1 or 2 with smaller total weight, a contradiction.

Now suppose $j_1 = j_2$. Denote this number by j . Then $V \cap F_{\{j, h_1, h_2\}}$ contains the vectors

$$(11.4) \quad e_j + e_{h_1} \text{ and } e_j + e_{h_2} \in V.$$

Since we are assuming that $m \geq 4$, $\{j, h_1, h_2\} \subsetneq \{1, \dots, m\}$ and hence, $V \cap F_{\{j, h_1, h_2\}}$ is almost coordinate. The subspace in $F_{\{j, h_1, h_2\}}$ generated by the two vectors (11.4) is cut by the linear equation

$$x_j + x_{h_1} + x_{h_2} = 0$$

and clearly is not almost coordinate. It follows that $V \cap F_{\{j, h_1, h_2\}} = F_{\{j, h_1, h_2\}}$, hence V contains all three of the coordinate vectors e_j, e_{h_1} and e_{h_2} . Replacing $e_j + e_{h_1}$ and $e_j + e_{h_2}$ by e_j, e_{h_1} and e_{h_2} in our spanning set, we reduce the total weight by one, a contradiction. This completes the proof of the claim and thus of Proposition 11.6. \square

12. COORDINATE SUBSPACES AND QUASI-PERMUTATION LATTICES

Proposition 12.1. *Let W be a finite group, M be a W -lattice and let $\lambda: M \rightarrow F := \mathbb{Z}/p\mathbb{Z}$ be a surjective morphism of W -modules, where p is a prime and W acts trivially on F . For any $m \geq 1$, and an F -subspace $S \subset V := F^m$, let M_S^m be the preimage of $S \subset F^m$ under the projection $\lambda^m: M^m \rightarrow F^m$.*

Assume that

- (a) *M is a quasi-permutation W -lattice;*
- (b) *the W^m -lattice $M_{S_1}^m$ is not quasi-permutation for any 1-dimensional subspace S_1 of F^m of the form $S_1 = \text{Span}_F\{(a_1, \dots, a_n)\}$, where $a_1 \neq 0, \dots, a_m \neq 0$.*

Then, given a subspace $S \subset F^m$, M_S^m is a quasi-permutation W^m -lattice if and only if S is coordinate.

The following notation will be helpful in the proof of Proposition 12.1 and in the subsequent sections.

Definition 12.2. Let W be a finite group, M be a W -module and m be a positive integer. Given a subset $I \subset \{1, \dots, m\}$, we define the “coordinate subgroup” $W_I \subset W^m$ as

$$W_I := \{(w_1, \dots, w_m) \mid w_i = 1 \text{ if } i \notin I\}.$$

We will also define the W_I -submodule M_I of M^m as

$$M_I := \{(a_1, \dots, a_m) \in M^m \mid a_i = 0 \text{ if } i \notin I\}.$$

We shorten $W_{\{i\}}$, $M_{\{i\}}$ to W_i , M_i .

Proof of Proposition 12.1. The “if” assertion is clear. We will prove “only if” by induction on m . In the base case, $m = 1$, every subspace of V is coordinate, so there is nothing to prove.

For the induction step, assume that $m \geq 2$ and that our assertion has been established for all $m' < m$. Suppose that for some subspace $S \subset F^m$ the lattice M_S^m is quasi-permutation. We want to show that S is coordinate.

Since M_S^m is quasi-permutation, Lemma 2.4 tells us that $M_S^m \cap M_I$ is a quasi-permutation W_I -lattice for every $I \subsetneq \{1, \dots, m\}$ (cf. Definition 12.2 above). But $M_S^m \cap M_I = M_{S \cap F_I}^m$, and so by the induction hypothesis $S \cap F_I$ is a coordinate subspace in F_I (and hence, in F^m).

Now Lemma 11.3 tells us that either S is a 1-dimensional subspace of F^m which does not lie in any coordinate hyperplane or S is a coordinate subspace in F^m . Our assumption (ii) rules out the first possibility. Hence, S is a coordinate subspace of F^m , as claimed. \square

13. PROOF OF THEOREM 7.1 FOR H OF TYPES \mathbf{A}_{n-1} , $n \geq 5$, \mathbf{B}_n ($n \geq 3$)
AND \mathbf{D}_n ($n \geq 4$)

13.1. Let R be an irreducible reduced root system. We denote by $Q = Q(R)$ the root lattice of R and by $P = P(R)$ the weight lattice of R , both lattices regarded as $W := W(R)$ -lattices. Given a positive integer m and a subset $I \subset \{1, \dots, m\}$, we define $W_I \subset W^m$, and the W_I -modules Q_I , P_I , etc., as in Definition 12.2. The base field k is assumed to be algebraically closed of characteristic zero.

Theorem 13.2. *Let $G = (\mathbf{SL}_n)^m/C$, where $n \geq 5$ and C is a subgroup of $(\mu_n)^m = Z(\mathbf{SL}_n^m)$. Then the following conditions are equivalent:*

- (a) G is Cayley,
- (b) G is stably Cayley,
- (c) the character lattice $X(G)$ is quasi-permutation,
- (d) $L = Q^m$,
- (e) G is isomorphic to $(\mathbf{PGL}_n)^m$.

Proof. (a) \implies (b) is obvious.

(b) \implies (c) follows from [LPR, Thm. 1.27].

(d) \implies (e): clear.

(e) \implies (a): clear, because the group \mathbf{PGL}_n is Cayley, see [LPR, Thm. 1.31], and a product of Cayley groups is obviously Cayley.

The implication (c) \implies (d) follows from the next proposition. \square

Proposition 13.3. *Let $R = \mathbf{A}_{n-1}$, where $n \geq 5$. Let $F = P/Q = \mathbb{Z}/n\mathbb{Z}$. Let L be an intermediate W^m -lattice between Q^m and P^m . If L is quasi-permutation, then $L = Q^m$.*

Proof. We proceed by induction on m . The base case, $m = 1$, follows from [LPR, Prop. 5.1]. For the induction step, assume that the proposition holds for $m - 1 \geq 1$. We show that it also holds for m .

We set $I = \{2, \dots, m\} \subset \{1, 2, \dots, m\}$. By Lemma 2.4, $L \cap P_I$ is a quasi-permutation W_I -lattice. By the induction hypothesis, $L \cap P_I = Q_I$. Set $S = L/Q^m \subset F^m$, then $S \cap F_I = 0$. It follows that the canonical projection $S \rightarrow F_1$ is injective. As $F = \mathbb{Z}/n\mathbb{Z}$, we have $S \cong \mathbb{Z}/d\mathbb{Z}$ for some divisor d of n .

In the notation of §10.1, $L = P_S^m$ as a W -lattice (where W acts on P^m diagonally). By Corollary 10.3,

$$(13.1) \quad L \cong L_1 \oplus Q^{m-1},$$

where $Q_1 \subset L_1 \subset P_1$. Clearly Q^{m-1} is quasi-permutation as a W -lattice because so is $Q = \ker[\mathbb{Z}[S_n/S_{n-1}] \rightarrow \mathbb{Z}]$. By assumption, L is a quasi-permutation W^m -lattice, hence it is quasi-permutation as a W -lattice. Since L and Q^{m-1} are quasi-permutation W -lattices, then from (13.1), we see that $L_1 \sim L_1 \oplus Q^{m-1} = L \sim 0$ so that L_1 is a quasi-permutation W -lattice. By [LPR, Prop. 5.1] it follows that $L_1 = Q_1$, hence $S = 0$, and $P_S^m = Q^m$. Thus

$L = Q^m$, which proves (d) for m and completes the proofs of Proposition 13.3 and Theorem 13.2. \square

13.4. Let $n \geq 7$ and R be the root system of \mathbf{Spin}_n (of type $\mathbf{B}_{(n-1)/2}$ for n odd or of type $\mathbf{D}_{n/2}$ for n even) and M be the character lattice of \mathbf{SO}_n . If n is odd, the $M = Q$; if n is even, then $Q \subsetneq M \subsetneq P$. Set $F := P/M \cong \mathbb{Z}/2\mathbb{Z}$.

Theorem 13.5. *Let $G = (\mathbf{Spin}_n)^m/C$, where $n \geq 7$, and C is a central subgroup of $(\mathbf{Spin}_n)^m$. Then the following conditions are equivalent:*

- (a) G is Cayley,
- (b) G is stably Cayley,
- (c) the character lattice $X(G)$ of G is quasi-permutation,
- (d) $X(G) = M^m$, where $M = X(\mathbf{SO}_n)$,
- (e) G is isomorphic to $(\mathbf{SO}_n)^m$.

Proof. Only (c) \implies (d) needs to be proved; the other implications are easy.

Assume (c), i.e., $X(G)$ is a quasi-permutation W^m -lattice. Clearly $Q^m \subset X(G) \subset P^m$. We claim that $X(G) \supset M^m$. If n is odd this is obvious, because $M^m = Q^m$. If n is even then by Lemma 2.4 $X(G) \cap P_i$ is a quasi-permutation W_i -lattice. Now by [LPR, Thm. 1.28] we have $L \cap P_i = M_i$. Thus $X(G) \supset M_1 \oplus \cdots \oplus M_m = M^m$, as claimed.

We will now show that $X(G) = M^m$. Assume the contrary. Consider the surjection $\lambda: P \rightarrow P/M \cong \mathbb{Z}/2\mathbb{Z}$. Set $S = X(G)/M^m \subset (\mathbb{Z}/2\mathbb{Z})^m$, then $S \neq 0$. In the notations of Lemma 10.2 we have $X(G) = P_S^m$. Since $S \neq 0$, by Corollary 10.4, $X(G)$ has a direct W -summand isomorphic to P . By Proposition 9.2, P is not quasi-invertible, hence $X(G)$ is not quasi-invertible as a W -lattice. It follows that $X(G)$ is not a quasi-invertible W^m -lattice, which contradicts (c). This contradiction shows that $X(G) = M^m$, which proves (d).

Alternatively, we can proceed, as in the proof of Proposition 13.3, to prove by induction that $X(G) = M^m$ using Corollary 10.3. Here we make use of the fact that by Proposition 9.2, P is not quasi-permutation. \square

Remark 13.6. Proposition 13.3 cannot be proved by an argument analogous to the first proof of Theorem 13.5. Indeed, the first proof of Theorem 13.5 relies on the fact that $X(\mathbf{Spin}_n)$ is not quasi-invertible for $n \geq 7$ (see Proposition 9.2). On the other hand, $X(\mathbf{SL}_n)$ is quasi-invertible whenever n is a prime; see [CS2, Prop. 9.1 and Rem. 9.3].

14. PROOF OF THEOREM 7.1 FOR H OF TYPE $\mathbf{A}_1 = \mathbf{B}_1 = \mathbf{C}_1$

We will continue to use the notations of §13.1. Let $R = \mathbf{A}_1$. Set $F = Q/P = \mathbb{Z}/2\mathbb{Z}$.

Let $G = (\mathbf{SL}_2)^m/C$, where C is a subgroup of $Z((\mathbf{SL}_2)^m) = (\mu_2)^m$. Clearly $Q^m \subset X(G) \subset P^m$, and $S := X(G)/Q^m \subset F^m = (\mathbb{Z}/2\mathbb{Z})^m$ is the orthogonal complement of $C \subset (\mu_2)^m$. Note that $F^m = (\mathbb{Z}/2\mathbb{Z})^m$ is the character group of $(\mu_2)^m$.

Theorem 14.1. *Let $G = (\mathbf{SL}_2)^m/C$, where C is a subgroup of $Z((\mathbf{SL}_2)^m) = (\mu_2)^m$. Then the following conditions are equivalent:*

- (a) G is Cayley,
- (b) G is stably Cayley,
- (c) the character lattice $X(G)$ is a quasi-permutation W^m -lattice,
- (d) $S := X(G)/Q^m$ is an almost coordinate subspace of $F^m = (\mathbb{Z}/2\mathbb{Z})^m$,
- (e) G decomposes into a direct product of normal subgroups $G_1 \times_k \cdots \times_k G_s$, where each G_i is isomorphic to either \mathbf{SL}_2 , \mathbf{PGL}_2 or \mathbf{SO}_4 .

Remark 14.2. The set of normal subgroups G_1, \dots, G_s in part (e) is uniquely determined by G ; see Remark 11.5.

Proof of Theorem 14.1. Only the implication (c) \implies (d) needs to be proved; all the other implications are easy. The implication (c) \implies (d) follows from the next proposition. \square

Proposition 14.3. *Let $R = \mathbf{A}_1$ and L be an intermediate W -lattice between Q^m and P^m , i.e., $Q^m \subset L \subset P^m$. Write $S = L/Q^m \subset F^m = (\mathbb{Z}/2\mathbb{Z})^m$. Then L is quasi-permutation if and only if S is almost coordinate.*

Proof. The “if” assertion follows easily from Lemmas 2.6 and 2.5. To prove the “only if” assertion, we begin by considering three special cases which will be of particular interest to us.

Case 1: $m \leq 2$. Here every subspace of $(\mathbb{Z}/2\mathbb{Z})^m$ is almost coordinate, and condition (d) holds automatically.

Case 2: S is the line $\langle \mathbf{1}_m \rangle = \{0, \mathbf{1}_m\} \subset (\mathbb{Z}/2\mathbb{Z})^m$, where $\mathbf{1}_m = \{1, \dots, 1\}$.

This $S = \langle \mathbf{1}_m \rangle$ is not almost coordinate for any $m \geq 3$. Thus we need to show that (c) does not hold, i.e., the lattice $L = P_{\langle \mathbf{1}_m \rangle}^m$ is not quasi-permutation. This lattice is isomorphic to the lattice M described in §9.1, in the case, where Δ is the disjoint union of m copies of \mathbf{B}_1 (or equivalently, of \mathbf{A}_1) for $m \geq 3$. By Proposition 9.2, for $m \geq 3$ the lattice $M \simeq L = P_{\langle \mathbf{1}_m \rangle}^m$, is not quasi-invertible, hence not quasi-permutation, as claimed.

Case 3: $m = 3$. There are two subspaces S of $(\mathbb{Z}/2\mathbb{Z})^3$ that are not almost coordinate: (i) the line $\langle \mathbf{1}_3 \rangle$ and (ii) the 2-dimensional subspace cut out by $x_1 + x_2 + x_3 = 0$. Once again we need to show that in both of these cases L is not quasi-permutation.

Case (i) is covered by Case 2 (with $m = 3$). If S is as in (ii), then L is isomorphic to the lattice M defined in the statement of Proposition 9.4. By this proposition, L is not quasi-invertible, hence not quasi-permutation, as claimed.

We now proceed with the proof of the proposition by induction on $m \geq 1$. The base case, where $m \leq 3$, is covered by Cases 1 and 3 above. For the induction step assume that $m \geq 4$ and that the proposition has been established for all $m' \leq m - 1$.

Suppose for some subspace $S = L/Q^m \subset (\mathbb{Z}/2\mathbb{Z})^m$ we know that $L = P_S^m$ is quasi-permutation. Our goal is to show that S is almost coordinate.

Since L is quasi-permutation, by Lemma 2.4 we conclude that $L \cap P_I$ is a quasi-permutation W_I -lattice for every $I = \{i_1, \dots, i_r\} \subsetneq \{1, \dots, m\}$. By the induction hypothesis $(L \cap P_I)/Q_I = S \cap F_I$ is an almost coordinate subspace in $F_I = (\mathbb{Z}/2\mathbb{Z})^r$.

Now Proposition 11.6 tells us that S is either the line $\langle \mathbf{1}_m \rangle$, or almost coordinate. If S is the line $\langle \mathbf{1}_m \rangle$, then L is not quasi-permutation by Case 2, contradicting our assumption. Thus S is almost coordinate, which completes the proofs of Proposition 14.3 and Theorem 14.1. \square

15. PROOF OF THEOREM 7.1 FOR H OF TYPES \mathbf{A}_2 , $\mathbf{B}_2 = \mathbf{C}_2$, AND $\mathbf{A}_3 = \mathbf{D}_3$

15.1. $R = \mathbf{A}_2$. Once again, we will continue to use the notations of §13.1. Set $F := P/Q \simeq \mathbb{Z}/3\mathbb{Z}$.

Theorem 15.1. *Let $G = (\mathbf{SL}_3)^m/C$, where C is a subgroup of $(\mu_3)^m = Z((\mathbf{SL}_3)^m)$. Then the following conditions are equivalent:*

- (a) G is Cayley,
- (b) G is stably Cayley,
- (c) the character lattice $X(G)$ is a quasi-permutation W^m -lattice,
- (d) $S := X(G)/Q^m$ is a coordinate subspace of $F^m \simeq (\mathbb{Z}/3\mathbb{Z})^m$,
- (e) G decomposes into a direct product of normal subgroups $G_1 \times_k \dots \times_k G_s$, where each G_i is isomorphic to either \mathbf{SL}_3 or \mathbf{PGL}_3 .

Proof. Only the implication (c) \implies (d) needs to be proved; the other implications are easy.

Clearly $Q^m \subset X(G) \subset P^m$; assume $X(G)$ is quasi-permutation. The W -lattices P and Q are quasi-permutation, see Theorem 1.4. If S is the 1-dimensional subspace $\langle \mathbf{a} \rangle$ spanned by a vector $\mathbf{a} = (a_1, \dots, a_m)$ such that $a_1 \neq 0, \dots, a_m \neq 0$, then from Proposition 9.9 it follows that $X(G) = P_{\langle \mathbf{a} \rangle}^m$ is not a quasi-permutation W^m -lattice, a contradiction. Now by Proposition 12.1, $X(G) = P_S^m$ is quasi-permutation if and only if S is coordinate. This shows that (c) \implies (d). \square

15.2. $R = \mathbf{B}_2 = \mathbf{C}_2$. Set $F := P/Q = \mathbb{Z}/2\mathbb{Z}$.

Theorem 15.2. *Let $G = (\mathbf{Spin}_5)^m/C$, where C is a subgroup of $(\mu_2)^m = \ker[(\mathbf{Spin}_5)^m \rightarrow (\mathbf{SO}_5)^m]$. Then the following conditions are equivalent:*

- (a) G is Cayley,
- (b) G is stably Cayley,
- (c) the character lattice $X(G)$ is quasi-permutation,
- (d) $S := X(G)/Q^m$ is a coordinate subspace of $F^m = (\mathbb{Z}/2\mathbb{Z})^m$,
- (e) G decomposes into a direct product of normal subgroups $G_1 \times_k \dots \times_k G_s$, where each G_i is isomorphic to either $\mathbf{Spin}_5 = \mathbf{Sp}_4$ or \mathbf{SO}_5 .

Proof. As in the proof of Theorem 15.1, we only need to establish the implication (c) \implies (d). We have $Q^m \subset X(G) \subset P^m$. The W -lattices P and

Q are quasi-permutation, see Theorem 1.4. If S is the 1-dimensional subspace $\langle \mathbf{1}_m \rangle$ spanned by the vector $\mathbf{1}_m = (1, \dots, 1)$ then by Proposition 9.2, $P_{\langle \mathbf{1}_m \rangle}^m$ is not a quasi-invertible W^m -lattice. Now by Proposition 12.1, the W^m -lattice $X(G) = P_S^m$ is quasi-permutation if and only if S is coordinate, which completes the proof of the theorem. \square

15.3. $R = \mathbf{A}_3 = \mathbf{D}_3$.

Theorem 15.3. *Let $G = (\mathbf{Spin}_6)^m / C'$, where C' is a subgroup of $(\mu_4)^m = \ker[(\mathbf{Spin}_6)^m \rightarrow (\mathbf{PSO}_6)^m]$. Clearly $Q^m \subset X(G) \subset P^m$, where P , Q and $X(G)$ are the character lattices of \mathbf{PSO}_6 , \mathbf{Spin}_6 and G , respectively. Then the following conditions are equivalent:*

- (a) G is Cayley,
- (b) G is stably Cayley,
- (c) $X(G)$ is quasi-permutation,
- (d) $X(G) \subset 2P^m$ and $X(G)/Q^m$ is a coordinate subspace of $(2P/Q)^m = (\mathbb{Z}/2\mathbb{Z})^m$,
- (e) G decomposes into a direct product of normal subgroups $G_1 \times_k \cdots \times_k G_s$, where each G_i is isomorphic to either \mathbf{SO}_6 or $\mathbf{PSO}_6 = \mathbf{PGL}_4$.

Proof. Both \mathbf{SO}_6 or $\mathbf{PSO}_6 = \mathbf{PGL}_4$ are Cayley; see, e.g., Theorem 1.4. Consequently, (e) \implies (a). Thus we only need to show that (c) \implies (d); the other implications are immediate. Assume that $X(G)$ is quasi-permutation.

First we claim that $X(G) \subset 2P^m$. Indeed, assume the contrary. Then $X(G)/Q^m$ contains an element of order 4. By Corollary 10.4 the W^m -lattice $X(G)$ restricted to the diagonal subgroup W has a direct summand isomorphic to the character lattice P of \mathbf{Spin}_6 . By Proposition 9.2 the W -lattice P is not quasi-invertible. We conclude that $X(G)$ is not a quasi-invertible as a W -lattice and hence not a quasi-invertible W^m -lattice, contradicting our assumption that $X(G)$ is quasi-permutation. This proves the claim.

As we mentioned above, \mathbf{SO}_6 and \mathbf{PSO}_6 are both Cayley. Hence, the W -lattices $2P$ and Q are quasi-permutation. If $X(G)/Q^m$ is the 1-dimensional subspace $\langle \mathbf{1}_m \rangle$ spanned by the vector $\mathbf{1}_m = (1, \dots, 1)$, then by Proposition 9.6, $X(G)$ is not a quasi-invertible W^m -lattice, a contradiction. Now Proposition 12.1 tells us that the W^m -lattice $X(G)/Q^m$ is coordinate in $(2P/Q)^m$ and (d) follows. \square

This completes the proof of Theorem 7.1.

16. PROOF OF THEOREM 1.5

In this section we deduce Theorem 1.5 from Theorem 7.1. Clearly (b) implies (a), so we only need to show that (a) implies (b).

Let G be a stably Cayley simple k -group. Let \bar{k} be a fixed algebraic closure of k , and set $\bar{G} = G \times_k \bar{k}$, then \bar{G} is a stably Cayley \bar{k} -group. Then by Theorem 7.1, $\bar{G} = G_1 \times_{\bar{k}} \cdots \times_{\bar{k}} G_s$, where each G_i is either a stably Cayley simple group or isomorphic to \mathbf{SO}_4 . If G is not of type \mathbf{A}_1 , then

each G_i is a simple group, and it follows that the decomposition into a direct product $\overline{G} = G_1 \times_{\bar{k}} G_2 \times_{\bar{k}} \cdots \times_{\bar{k}} G_s$ is uniquely determined by \overline{G} . If G is of type \mathbf{A}_1 , then by Remark 14.2 the decomposition into a direct product $\overline{G} = G_1 \times_{\bar{k}} \cdots \times_{\bar{k}} G_s$ is again uniquely determined by \overline{G} . The Galois group $\text{Gal}(\bar{k}/k)$ acts on \overline{G} , hence it acts (transitively) on the set of direct factors $\{G_i\}$. Set $G' = G_2 \times_{\bar{k}} \cdots \times_{\bar{k}} G_s$, then $\overline{G} = G_1 \times_{\bar{k}} G'$. Let $l \subset \bar{k}$ be the subfield corresponding to the stabilizer of G_1 in $\text{Gal}(\bar{k}/k)$, then G_1 and G' are $\text{Gal}(\bar{k}/l)$ -invariant, and we obtain l -forms of these two \bar{k} -groups, which, by abuse of notation, we will again call G_1 and G' . Then $G = R_{l/k}G_1$ and $G_l = G_1 \times_l G'$. Now, since G is stably Cayley over k , we see that G_l is stably Cayley over l . It follows from construction that G_1 is either an absolutely simple group or an l -form of \mathbf{SO}_4 . In the latter case G_1 is an *outer* l -form of \mathbf{SO}_4 (otherwise G_1 is not l -simple and G is not k -simple).

We wish to prove that G_1 is stably Cayley over l . Note that G_1 is a direct factor of G_l . However, we cannot use [LPR, Lemma 4.7] in order to conclude that G_1 is stably Cayley over l , because the proof of this lemma does not go through over non-closed fields. Instead, we use Theorem 1.4. If G_1 is stably Cayley over \bar{k} , but is not stably Cayley over l , then, comparing Theorem 1.4 and [LPR, Thm. 1.28], we see that G_1 is an outer l -form of \mathbf{PGL}_{2n} for some even number $2n \geq 4$.

We show that if G_1 is an outer l -form of \mathbf{PGL}_{2n} for some even number $2n \geq 4$, then $G := R_{l/k}G_1$ is not stably Cayley over k . It suffices to prove that $G_l = G_1 \times_l G'$ is not stably Cayley over l . Choose a maximal torus $T = T_1 \times_l T'$ of G_l . Set $\overline{T} := T \times_l \bar{k} = \overline{T}_1 \times_{\bar{k}} \overline{T}'$.

Let $L_1 := \mathbf{X}(\overline{T}_1)$ and $L' := \mathbf{X}(\overline{T}')$ denote the corresponding character lattices, and let W_1 and W' be the corresponding Weyl groups. Choose a Borel subgroup $\overline{B} \subset \overline{G}$ containing \overline{T} . Let A_Ψ denote the image of the Galois group $\text{Gal}(\bar{k}/l)$ in $\text{Aut}(L_1) \times \text{Aut}(L')$ with respect to the $*$ -action. By Theorem 1.3, it suffices to show that the $(W_1 \times W') \rtimes A_\Psi$ -lattice $L_1 \oplus L'$ is not quasi-permutation.

Denote by V_1 the image of the group $V := (W_1 \times W') \rtimes A_\Psi$ in $\text{Aut}(L_1)$. Since G_1 is an outer form over l , we see that $V_1 = \text{Aut}(\mathbf{A}_{2n-1})$. Note that the $\text{Aut}(\mathbf{A}_{2n-1})$ -lattice L_1 is isomorphic to $\mathbb{Z}\mathbf{A}_{2n-1}$. In [CK, §5.1] it was proved that there exist a subgroup $\Gamma \subset \text{Aut}(\mathbf{A}_{2n-1})$ isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ and a direct summand M of the Γ -lattice $\mathbb{Z}\mathbf{A}_{2n-1}$ isomorphic to J_Γ . Let $\Delta \subset V = (W_1 \times W') \rtimes A_\Psi$ be the preimage of $\Gamma \subset \text{Aut}(\mathbf{A}_{2n-1}) = V_1$. By Corollary 8.7, we have $\text{III}^2(\Delta, M) \neq 0$, and therefore M is not a quasi-invertible Δ -lattice. It follows that L_1 is not a quasi-invertible Δ -lattice, hence L_1 is not a quasi-invertible $(W_1 \times W') \rtimes A_\Psi$ -lattice, hence $L_1 \oplus L'$ is not a quasi-permutation $(W_1 \times W') \rtimes A_\Psi$ -lattice. Thus by Theorem 1.3 G_l is not stably Cayley over l , and therefore G is not stably Cayley over k . This completes the proof of Theorem 1.5. \square

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